

## CHAPTER

## 11

Limits, Continuity and  
Differentiability

## Section-A

## JEE Advanced/ IIT-JEE

## A Fill in the Blanks

1. Let  $f(x) = \begin{cases} (x-1)^2 \sin \frac{1}{(x-1)} |x| & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$   
be a real-valued function. Then the set of points where  $f(x)$  is not differentiable is ..... (1981 - 2 Marks)

2. Let  $f(x) = \begin{cases} \frac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$   
If  $f(x)$  is continuous for all  $x$ , then  $k = \dots\dots\dots$  (1981 - 2 Marks)

3. A discontinuous function  $y = f(x)$  satisfying  $x^2 + y^2 = 4$  is given by  $f(x) = \dots\dots\dots$  (1982 - 2 Marks)

4.  $\lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2} = \dots\dots\dots$  (1984 - 2 Marks)

5. If  $f(x) = \sin x$ ,  $x \neq n\pi$ ,  $n = 0, \pm 1, \pm 2, \pm 3, \dots\dots\dots$   
 $= 2$ , otherwise

and  $g(x) = x^2 + 1$ ,  $x \neq 0, 2$   
 $= 4$ ,  $x = 0$   
 $= 5$ ,  $x = 2$ ,

then  $\lim_{x \rightarrow 0} g[f(x)]$  is ..... (1986 - 2 Marks)

6.  $\lim_{x \rightarrow -\infty} \left[ \frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{(1 + |x|^3)} \right] = \dots\dots\dots$  (1987 - 2 Marks)

7. If  $f(9) = 9$ ,  $f'(9) = 4$ , then  $\lim_{x \rightarrow 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$  equals ..... (1988 - 2 Marks)

8.  $ABC$  is an isosceles triangle inscribed in a circle of radius  $r$ . If  $AB = AC$  and  $h$  is the altitude from  $A$  to  $BC$  then the triangle  $ABC$  has perimeter  $P = 2(\sqrt{2hr - h^2}) + \sqrt{2hr}$  and area  $A = \dots\dots\dots$  also  $\lim_{h \rightarrow 0} \frac{A}{P^3} = \dots\dots\dots$  (1989 - 2 Marks)

9.  $\lim_{x \rightarrow \infty} \left( \frac{x+6}{x+1} \right)^{x+4} = \dots\dots\dots$  (1990 - 2 Marks)

10. Let  $f(x) = x |x|$ . The set of points where  $f(x)$  is twice differentiable is ..... (1992 - 2 Marks)

11. Let  $f(x) = [x] \sin \left( \frac{\pi}{[x+1]} \right)$ , where  $[\bullet]$  denotes the greatest integer function. The domain of  $f$  is... and the points of discontinuity of  $f$  in the domain are.... (1996 - 2 Marks)

12.  $\lim_{x \rightarrow 0} \left( \frac{1+5x^2}{1+3x^2} \right)^{1/x^2} = \dots\dots\dots$  (1996 - 1 Mark)

13. Let  $f(x)$  be a continuous function defined for  $1 \leq x \leq 3$ . If  $f(x)$  takes rational values for all  $x$  and  $f(2) = 10$ , then  $f(1.5) = \dots\dots\dots$  (1997 - 2 Marks)

## B True / False

1. If  $\lim_{x \rightarrow a} [f(x)g(x)]$  exists then both  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. (1981 - 2 Marks)

## C MCQs with One Correct Answer

1. If  $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$ , then  $\lim_{x \rightarrow \infty} f(x)$  is ..... (1979)

- (a) 0 (b)  $\infty$   
(c) 1 (d) none of these

2. For a real number  $y$ , let  $[y]$  denotes the greatest integer less than or equal to  $y$ : Then the function  $f(x) = \frac{\tan(\pi[x - \pi])}{1 + [x]^2}$  is ..... (1981 - 2 Marks)

- (a) discontinuous at some  $x$   
(b) continuous at all  $x$ , but the derivative  $f'(x)$  does not exist for some  $x$   
(c)  $f'(x)$  exists for all  $x$ , but the second derivative  $f''(x)$  does not exist for some  $x$   
(d)  $f'(x)$  exists for all  $x$

3. There exist a function  $f(x)$ , satisfying  $f(0) = 1, f'(0) = -1, f(x) > 0$  for all  $x$ , and (1982 - 2 Marks)
- (a)  $f''(x) > 0$  for all  $x$  (b)  $-1 < f''(x) < 0$  for all  $x$   
 (c)  $-2 \leq f''(x) \leq -1$  for all  $x$  (d)  $f''(x) < -2$  for all  $x$
4. If  $G(x) = -\sqrt{25-x^2}$  then  $\lim_{x \rightarrow 1} \frac{G(x)-G(1)}{x-1}$  has the value (1983 - 1 Mark)
- (a)  $\frac{1}{24}$  (b)  $\frac{1}{5}$   
 (c)  $-\sqrt{24}$  (d) none of these
5. If  $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2$ , then the value of  $\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x-a}$  is (1983 - 1 Mark)
- (a)  $-5$  (b)  $\frac{1}{5}$   
 (c)  $5$  (d) none of these
6. The function  $f(x) = \frac{\ln(1+ax) - \ln(1-bx)}{x}$  is not defined at  $x = 0$ . The value which should be assigned to  $f$  at  $x = 0$  so that it is continuous at  $x = 0$ , is (1983 - 1 Mark)
- (a)  $a - b$  (b)  $a + b$   
 (c)  $\ln a - \ln b$  (d) none of these
7.  $\lim_{n \rightarrow \infty} \left\{ \frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right\}$  is equal to (1984 - 2 Marks)
- (a)  $0$  (b)  $-\frac{1}{2}$   
 (c)  $\frac{1}{2}$  (d) none of these
8. If  $f(x) = \frac{\sin[x]}{[x]}$ ,  $[x] \neq 0$  (1985 - 2 Marks)  
 $= 0, [x] = 0$
- Where  $[x]$  denotes the greatest integer less than or equal to  $x$ . then  $\lim_{x \rightarrow 0} f(x)$  equals -
- (a)  $1$  (b)  $0$   
 (c)  $-1$  (d) none of these
9. Let  $f: R \rightarrow R$  be a differentiable function and  $f(1) = 4$ . Then the value of  $\lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} dt$  is (1990 - 2 Marks)
- (a)  $8f'(1)$  (b)  $4f'(1)$  (c)  $2f'(1)$  (d)  $f'(1)$
10. Let  $[.]$  denote the greatest integer function and  $f(x) = [\tan^2 x]$ , then: (1993 - 1 Mark)
- (a)  $\lim_{x \rightarrow 0} f(x)$  does not exist  
 (b)  $f(x)$  is continuous at  $x = 0$   
 (c)  $f(x)$  is not differentiable at  $x = 0$   
 (d)  $f'(0) = 1$
11. The function  $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$ ,  $[.]$  denotes the greatest integer function, is discontinuous at (1995S)
- (a) All  $x$  (b) All integer points  
 (c) No  $x$  (d)  $x$  which is not an integer
12.  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$  equals (1997 - 2 Marks)
- (a)  $1 + \sqrt{5}$  (b)  $-1 + \sqrt{5}$  (c)  $-1 + \sqrt{2}$  (d)  $1 + \sqrt{2}$
13. The function  $f(x) = [x]^2 - [x^2]$  (where  $[y]$  is the greatest integer less than or equal to  $y$ ), is discontinuous at (1999 - 2 Marks)
- (a) all integers  
 (b) all integers except 0 and 1  
 (c) all integers except 0  
 (d) all integers except 1
14. The function  $f(x) = (x^2 - 1) |x^2 - 3x + 2| + \cos(|x|)$  is NOT differentiable at (1999 - 2 Marks)
- (a)  $-1$  (b)  $0$  (c)  $1$  (d)  $2$
15.  $\lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$  is (1999 - 2 Marks)
- (a)  $2$  (b)  $-2$  (c)  $1/2$  (d)  $-1/2$
16. For  $x \in R, \lim_{x \rightarrow \infty} \left( \frac{x-3}{x+2} \right)^x =$  (2000S)
- (a)  $e$  (b)  $e^{-1}$  (c)  $e^{-5}$  (d)  $e^5$
17.  $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$  equals (2001S)
- (a)  $-\pi$  (b)  $\pi$  (c)  $\pi/2$  (d)  $1$
18. The left-hand derivative of  $f(x) = [x] \sin(\pi x)$  at  $x = k, k$  an integer, is (2001S)
- (a)  $(-1)^k(k-1)\pi$  (b)  $(-1)^{k-1}(k-1)\pi$   
 (c)  $(-1)^k k\pi$  (d)  $(-1)^{k-1} k\pi$
19. Let  $f: R \rightarrow R$  be a function defined by  $f(x) = \max\{x, x^3\}$ . The set of all points where  $f(x)$  is NOT differentiable is (2001S)
- (a)  $\{-1, 1\}$  (b)  $\{-1, 0\}$  (c)  $\{0, 1\}$  (d)  $\{-1, 0, 1\}$
20. Which of the following functions is differentiable at  $x = 0$ ? (2001S)
- (a)  $\cos(|x|) + |x|$  (b)  $\cos(|x|) - |x|$   
 (c)  $\sin(|x|) + |x|$  (d)  $\sin(|x|) - |x|$
21. The domain of the derivative of the function
- $$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \leq 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$
- is (2002S)
- (a)  $R - \{0\}$  (b)  $R - \{1\}$   
 (c)  $R - \{-1\}$  (d)  $R - \{-1, 1\}$

## Limits, Continuity and Differentiability

22. The integer  $n$  for which  $\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$  is a finite non-zero number is (2002S)  
 (a) 1 (b) 2 (c) 3 (d) 4
23. Let  $f: R \rightarrow R$  be such that  $f(1) = 3$  and  $f'(1) = 6$ . Then  $\lim_{x \rightarrow 0} \left( \frac{f(1+x)}{f(1)} \right)^{1/x}$  equals (2002S)  
 (a) 1 (b)  $e^{1/2}$  (c)  $e^2$  (d)  $e^3$
24. If  $\lim_{x \rightarrow 0} \frac{((a-n)nx - \tan x)\sin nx}{x^2} = 0$ , where  $n$  is nonzero real number, then  $a$  is equal to (2003S)  
 (a) 0 (b)  $\frac{n+1}{n}$  (c)  $n$  (d)  $n + \frac{1}{n}$
25.  $\lim_{h \rightarrow 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)}$ , given that  $f'(2) = 6$  and  $f'(1) = 4$   
 (a) does not exist (b) is equal to  $-3/2$   
 (c) is equal to  $3/2$  (d) is equal to 3 (2003S)
26. If  $f(x)$  is differentiable and strictly increasing function, then the value of  $\lim_{x \rightarrow 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$  is (2004S)  
 (a) 1 (b) 0 (c)  $-1$  (d) 2
27. The function given by  $y = ||x| - 1|$  is differentiable for all real numbers except the points (2005S)  
 (a)  $\{0, 1, -1\}$  (b)  $\pm 1$  (c) 1 (d)  $-1$
28. If  $f(x)$  is continuous and differentiable function and  $f(1/n) = 0 \forall n \geq 1$  and  $n \in I$ , then (2005S)  
 (a)  $f(x) = 0, x \in (0, 1]$   
 (b)  $f(0) = 0, f'(0) = 0$   
 (c)  $f(0) = 0 = f'(0), x \in (0, 1]$   
 (d)  $f(0) = 0$  and  $f'(0)$  need not to be zero
29. The value of  $\lim_{x \rightarrow 0} \left( (\sin x)^{1/x} + (1+x)^{\sin x} \right)$ , where  $x > 0$  is (2006 - 3M, -1)  
 (a) 0 (b)  $-1$  (c) 1 (d) 2
30. Let  $f(x)$  be differentiable on the interval  $(0, \infty)$  such that  $f(1) = 1$ , and  $\lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$  for each  $x > 0$ . Then  $f(x)$  is (2007 - 3 marks)  
 (a)  $\frac{1}{3x} + \frac{2x^2}{3}$  (b)  $\frac{-1}{3x} + \frac{4x^2}{3}$  (c)  $\frac{-1}{x} + \frac{2}{x^2}$  (d)  $\frac{1}{x}$
31.  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec^2 x}{x^2 - \frac{\pi^2}{16}}$  equals (2007 - 3 marks)  
 (a)  $\frac{8}{\pi} f(2)$  (b)  $\frac{2}{\pi} f(2)$  (c)  $\frac{2}{\pi} f\left(\frac{1}{2}\right)$  (d)  $4f(2)$
32. Let  $g(x) = \frac{(x-1)^n}{\log \cos^m(x-1)}$ ;  $0 < x < 2$ ,  $m$  and  $n$  are integers,  $m \neq 0, n > 0$ , and let  $p$  be the left hand derivative of  $|x-1|$  at  $x=1$ . If  $\lim_{x \rightarrow 1^+} g(x) = p$ , then (2008)  
 (a)  $n=1, m=1$  (b)  $n=1, m=-1$   
 (c)  $n=2, m=2$  (d)  $n > 2, m=n$
33. If  $\lim_{x \rightarrow 0} \left[ 1 + x \ln(1+b^2) \right]^{1/x} = 2b \sin^2 \theta, b > 0$  and  $\theta \in (-\pi, \pi]$ , then the value of  $\theta$  is (2011)  
 (a)  $\pm \frac{\pi}{4}$  (b)  $\pm \frac{\pi}{3}$  (c)  $\pm \frac{\pi}{6}$  (d)  $\pm \frac{\pi}{2}$
34. If  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$ , then (2012)  
 (a)  $a=1, b=4$  (b)  $a=1, b=-4$   
 (c)  $a=2, b=-3$  (d)  $a=2, b=3$
35. Let  $f(x) = \begin{cases} x^2 \left| \cos \frac{\pi}{x} \right|, & x \neq 0 \\ 0, & x = 0 \end{cases}, x \in R$  then  $f$  is (2012)  
 (a) differentiable both at  $x=0$  and at  $x=2$   
 (b) differentiable at  $x=0$  but not differentiable at  $x=2$   
 (c) not differentiable at  $x=0$  but differentiable at  $x=2$   
 (d) differentiable neither at  $x=0$  nor at  $x=2$
36. Let  $\alpha(a)$  and  $\beta(a)$  be the roots of the equation  $(\sqrt[3]{1+a}-1)x^2 + (\sqrt{1+a}-1)x + (\sqrt[6]{1+a}-1) = 0$  where  $a > -1$ . Then  $\lim_{a \rightarrow 0^+} \alpha(a)$  and  $\lim_{a \rightarrow 0^+} \beta(a)$  are (2012)  
 (a)  $-\frac{5}{2}$  and 1 (b)  $-\frac{1}{2}$  and  $-1$   
 (c)  $-\frac{7}{2}$  and 2 (d)  $-\frac{9}{2}$  and 3

**D MCQs with One or More than One Correct**

1. If  $x + |y| = 2y$ , then  $y$  as a function of  $x$  is (1984 - 3 Marks)  
 (a) defined for all real  $x$   
 (b) continuous at  $x=0$   
 (c) differentiable for all  $x$   
 (d) such that  $\frac{dy}{dx} = \frac{1}{3}$  for  $x < 0$
2. If  $f(x) = x(\sqrt{x} - \sqrt{x+1})$ , then— (1985 - 2 Marks)  
 (a)  $f(x)$  is continuous but not differentiable at  $x=0$   
 (b)  $f(x)$  is differentiable at  $x=0$   
 (c)  $f(x)$  is not differentiable at  $x=0$   
 (d) none of these

3. The function  $f(x) = 1 + |\sin x|$  is (1986 - 2 Marks)  
 (a) continuous nowhere  
 (b) continuous everywhere  
 (c) differentiable nowhere  
 (d) not differentiable at  $x = 0$   
 (e) not differentiable at infinite number of points.
4. Let  $[x]$  denote the greatest integer less than or equal to  $x$ . If  $f(x) = [x \sin \pi x]$ , then  $f(x)$  is (1986 - 2 Marks)  
 (a) continuous at  $x = 0$  (b) continuous in  $(-1, 0)$   
 (c) differentiable at  $x = 1$  (d) differentiable in  $(-1, 1)$   
 (e) none of these
5. The set of all points where the function  $f(x) = \frac{x}{(1+|x|)}$  is differentiable, is (1987 - 2 Marks)  
 (a)  $(-\infty, \infty)$  (b)  $[0, \infty)$   
 (c)  $(-\infty, 0) \cup (0, \infty)$  (d)  $(0, \infty)$   
 (e) None
6. The function  $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$  is (1988 - 2 Marks)  
 (a) continuous at  $x = 1$  (b) differentiable at  $x = 1$   
 (c) continuous at  $x = 3$  (d) differentiable at  $x = 3$ .
7. If  $f(x) = \frac{1}{2}x - 1$ , then on the interval  $[0, \pi]$  (1989 - 2 Marks)  
 (a)  $\tan [f(x)]$  and  $1/f(x)$  are both continuous  
 (b)  $\tan [f(x)]$  and  $1/f(x)$  are both discontinuous  
 (c)  $\tan [f(x)]$  and  $f^{-1}(x)$  are both continuous  
 (d)  $\tan [f(x)]$  is continuous but  $1/f(x)$  is not.
8. The value of  $\lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x}$  (1991 - 2 Marks)  
 (a) 1 (b) -1  
 (c) 0 (d) none of these
9. The following functions are continuous on  $(0, \pi)$ . (1991 - 2 Marks)  
 (a)  $\tan x$   
 (b)  $\int_0^x t \sin \frac{1}{t} dt$   
 (c)  $\begin{cases} 1, & 0 < x \leq \frac{3\pi}{4} \\ 2 \sin \frac{2}{9}x, & \frac{3\pi}{4} < x < \pi \end{cases}$   
 (d)  $\begin{cases} x \sin x, & 0 < x \leq \frac{\pi}{2} \\ \frac{\pi}{2} \sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$
10. Let  $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$  then for all  $x$  (1994)  
 (a)  $f'$  is differentiable (b)  $f$  is differentiable  
 (c)  $f'$  is continuous (d)  $f$  is continuous
11. Let  $g(x) = xf(x)$ , where  $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ . At  $x = 0$   
 (a)  $g$  is differentiable but  $g'$  is not continuous (1994)  
 (b)  $g$  is differentiable while  $f$  is not  
 (c) both  $f$  and  $g$  are differentiable  
 (d)  $g$  is differentiable and  $g'$  is continuous
12. The function  $f(x) = \max \{(1-x), (1+x), 2\}$ ,  $x \in (-\infty, \infty)$  is (1995)  
 (a) continuous at all points  
 (b) differentiable at all points  
 (c) differentiable at all points except at  $x = 1$  and  $x = -1$   
 (d) continuous at all points except at  $x = 1$  and  $x = -1$ , where it is discontinuous
13. Let  $h(x) = \min \{x, x^2\}$ , for every real number of  $x$ , Then (1998 - 2 Marks)  
 (a)  $h$  is continuous for all  $x$   
 (b)  $h$  is differentiable for all  $x$   
 (c)  $h'(x) = 1$ , for all  $x > 1$   
 (d)  $h$  is not differentiable at two values of  $x$ .
14.  $\lim_{x \rightarrow 1} \frac{\sqrt{1 - \cos 2(x-1)}}{x-1}$  (1998 - 2 Marks)  
 (a) exists and it equals  $\sqrt{2}$   
 (b) exists and it equals  $-\sqrt{2}$   
 (c) does not exist because  $x-1 \rightarrow 0$   
 (d) does not exist because the left hand limit is not equal to the right hand limit.
15. If  $f(x) = \min \{1, x^2, x^3\}$ , then (2006 - 5M, -1)  
 (a)  $f(x)$  is continuous  $\forall x \in \mathbf{R}$   
 (b)  $f(x)$  is continuous and differentiable everywhere.  
 (c)  $f(x)$  is not differentiable at two points  
 (d)  $f(x)$  is not differentiable at one point
16. Let  $L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$ ,  $a > 0$ .  
 If  $L$  is finite, then (2009)  
 (a)  $a = 2$  (b)  $a = 1$  (c)  $L = \frac{1}{64}$  (d)  $L = \frac{1}{32}$
17. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function such that  $f(x+y) = f(x) + f(y)$ ,  $\forall x, y \in \mathbf{R}$ . If  $f(x)$  is differentiable at  $x = 0$ , then (2011)  
 (a)  $f(x)$  is differentiable only in a finite interval containing zero  
 (b)  $f(x)$  is continuous  $\forall x \in \mathbf{R}$   
 (c)  $f'(x)$  is constant  $\forall x \in \mathbf{R}$   
 (d)  $f(x)$  is differentiable except at finitely many points.

$$18. \text{ If } f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \leq 0, \\ x-1, & 0 < x \leq 1 \\ \ln x, & x > 1 \end{cases} \quad (2011)$$

- (a)  $f(x)$  is continuous at  $x = -\frac{\pi}{2}$   
 (b)  $f(x)$  is not differentiable at  $x = 0$   
 (c)  $f(x)$  is differentiable at  $x = 1$   
 (d)  $f(x)$  is differentiable at  $x = -\frac{3}{2}$

19. For every integer  $n$ , let  $a_n$  and  $b_n$  be real numbers. Let function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be given by (2012)

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}$$

for all integers  $n$ . If  $f$  is continuous, then which of the following hold(s) for all  $n$ ?

- (a)  $a_{n-1} - b_{n-1} = 0$       (b)  $a_n - b_n = 1$   
 (c)  $a_n - b_{n+1} = 1$       (d)  $a_{n-1} - b_n = -1$

20. For  $a \in \mathbb{R}$  (the set of all real numbers),  $a \neq -1$ , (JEE Adv. 2013)

$$\lim_{n \rightarrow \infty} \frac{(1^a + 2^a + \dots + n^a)}{(n+1)^{a-1} [(na+1) + (na+2) + \dots + (na+n)]} = \frac{1}{60}$$

Then  $a =$

- (a) 5      (b) 7      (c)  $\frac{-15}{2}$       (d)  $\frac{-17}{2}$

21. Let  $f: [a, b] \rightarrow [1, \infty)$  be a continuous function and let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined as (JEE Adv. 2014)

$$g(x) = \begin{cases} 0, & \text{if } x < a, \\ \int_a^x f(t) dt, & \text{if } a \leq x \leq b; \\ \int_a^b f(t) dt, & \text{if } x > b. \end{cases}$$

- (a)  $g(x)$  is continuous but not differentiable at  $a$   
 (b)  $g(x)$  is differentiable on  $\mathbb{R}$   
 (c)  $g(x)$  is continuous but not differentiable at  $b$   
 (d)  $g(x)$  is continuous and differentiable at either (a) or (b) but not both

22. For every pair of continuous functions  $f, g: [0, 1] \rightarrow \mathbb{R}$  such that  $\max \{f(x) : x \in [0, 1]\} = \max \{g(x) : x \in [0, 1]\}$ , the correct statement(s) is (are): (JEE Adv. 2014)

- (a)  $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$  for some  $c \in [0, 1]$   
 (b)  $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$  for some  $c \in [0, 1]$   
 (c)  $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$  for some  $c \in [0, 1]$   
 (d)  $(f(c))^2 = (g(c))^2$  for some  $c \in [0, 1]$

23. Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function with  $g(0) = 0$ ,

$$g'(0) = 0 \text{ and } g'(1) \neq 0. \text{ Let } f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and  $h(x) = e^{|x|}$  for all  $x \in \mathbb{R}$ . Let  $(f \circ h)(x)$  denote  $f(h(x))$  and  $(h \circ f)(x)$  denote  $h(f(x))$ . Then which of the following is (are) true? (JEE Adv. 2015)

- (a)  $f$  is differentiable at  $x = 0$   
 (b)  $h$  is differentiable at  $x = 0$   
 (c)  $f \circ h$  is differentiable at  $x = 0$   
 (d)  $h \circ f$  is differentiable at  $x = 0$

24. Let  $a, b \in \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$ . Then  $f$  is (JEE Adv. 2016)

- (a) differentiable at  $x=0$  if  $a=0$  and  $b=1$   
 (b) differentiable at  $x=1$  if  $a=1$  and  $b=0$   
 (c) NOT differentiable at  $x=0$  if  $a=1$   $b=0$   
 (d) NOT differentiable at  $x=1$  if  $a=1$  and  $b=1$

25. Let  $f: \left[-\frac{1}{2}, 2\right] \rightarrow \mathbb{R}$  and  $g: \left[-\frac{1}{2}, 2\right] \rightarrow \mathbb{R}$  be functions defined by  $f(x) = [x^2 - 3]$  and  $g(x) = |x|f(x) + |4x - 7|f(x)$ , where  $[y]$  denotes the greatest integer less than or equal to  $y$  for  $y \in \mathbb{R}$ . Then (JEE Adv. 2016)

- (a)  $f$  is discontinuous exactly at three points in  $\left[-\frac{1}{2}, 2\right]$   
 (b)  $f$  is discontinuous exactly at four points in  $\left[-\frac{1}{2}, 2\right]$   
 (c)  $g$  is NOT differentiable exactly at four points in  $\left(-\frac{1}{2}, 2\right)$   
 (d)  $g$  is NOT differentiable exactly at five points in  $\left(-\frac{1}{2}, 2\right)$

## E Subjective Problems

1. Evaluate  $\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$ , ( $a \neq 0$ ) (1978)
2.  $f(x)$  is the integral of  $\frac{2 \sin x - \sin 2x}{x^3}$ ,  $x \neq 0$ , find  $\lim_{x \rightarrow 0} f'(x)$  (1979)

3. Evaluate:  $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$  (1980)

4. Let  $f(x+y) = f(x) + f(y)$  for all  $x$  and  $y$ . If the function  $f(x)$  is continuous at  $x = 0$ , then show that  $f(x)$  is continuous at all  $x$ . (1981 - 2 Marks)

5. Use the formula  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$  to find

$$\lim_{x \rightarrow 0} \frac{2^x - 1}{(1+x)^{1/2} - 1} \quad (1982 - 2 \text{ Marks})$$

6. Let  $f(x) = \begin{cases} 1+x, & 0 \leq x \leq 2 \\ 3-x, & 2 \leq x \leq 3 \end{cases}$  (1983 - 2 Marks)

Determine the form of  $g(x) = f[f(x)]$  and hence find the points of discontinuity of  $g$ , if any

7. Let  $f(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x < 1 \\ 2x^2 - 3x + \frac{3}{2}, & 1 \leq x \leq 2 \end{cases}$  (1983 - 2 Marks)

Discuss the continuity of  $f, f'$  and  $f''$  on  $[0, 2]$ .

8. Let  $f(x) = x^3 - x^2 + x + 1$  and  $g(x) = \max\{f(t); 0 \leq t \leq x\}$ ,  $0 \leq x \leq 1$  (1985 - 5 Marks)  
 $= 3 - x$   $0 \leq x \leq 2$

Discuss the continuity and differentiability of the function  $g(x)$  in the interval  $(0, 2)$ .

9. Let  $f(x)$  be defined in the interval  $[-2, 2]$  such that

$$f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$$

and  $g(x) = f(|x|) + |f(x)|$

Test the differentiability of  $g(x)$  in  $(-2, 2)$ . (1986 - 5 Marks)

10. Let  $f(x)$  be a continuous and  $g(x)$  be a discontinuous function. prove that  $f(x) + g(x)$  is a discontinuous function. (1987 - 2 Marks)

11. Let  $f(x)$  be a function satisfying the condition  $f(-x) = f(x)$  for all real  $x$ . If  $f'(0)$  exists, find its value. (1987 - 2 Marks)

12. Find the values of  $a$  and  $b$  so that the function

$$f(x) = \begin{cases} x + a\sqrt{2} \sin x, & 0 \leq x < \pi/4 \\ 2x \cot x + b, & \pi/4 \leq x \leq \pi/2 \\ a \cos 2x - b \sin x, & \pi/2 < x \leq \pi \end{cases}$$

is continuous for  $0 \leq x \leq \pi$ . (1989 - 2 Marks)

13. Draw a graph of the function  $y = [x] + |1 - x|$ ,  $-1 \leq x \leq 3$ . Determine the points, if any, where this function is not differentiable. (1989 - 4 Marks)

14. Let  $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & x > 0 \end{cases}$  (1990 - 4 Marks)

Determine the value of  $a$ , if possible, so that the function is continuous at  $x = 0$

15. A function  $f: R \rightarrow R$  satisfies the equation  $f(x+y) = f(x)f(y)$  for all  $x, y$  in  $R$  and  $f(x) \neq 0$  for any  $x$  in  $R$ . Let the function be differentiable at  $x = 0$  and  $f'(0) = 2$ . Show that  $f'(x) = 2f(x)$  for all  $x$  in  $R$ . Hence, determine  $f(x)$ . (1990 - 4 Marks)

16. Find  $\lim_{x \rightarrow 0} \{\tan(\pi/4 + x)\}^{1/x}$  (1993 - 2 Marks)

17. Let  $f(x) = \begin{cases} \{1 + |\sin x|\}^{a/|\sin x|} & ; \frac{\pi}{6} < x < 0 \\ b & ; x = 0 \\ e^{\tan 2x / \tan 3x} & ; 0 < x < \frac{\pi}{6} \end{cases}$

(1994 - 4 Marks)

Determine  $a$  and  $b$  such that  $f(x)$  is continuous at  $x = 0$

18. Let  $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$  for all real  $x$  and  $y$ . If  $f'(0)$

exists and equals  $-1$  and  $f(0) = 1$ , find  $f(2)$ . (1995 - 5 Marks)

19. Determine the values of  $x$  for which the following function fails to be continuous or differentiable: (1997 - 5 Marks)

$$f(x) = \begin{cases} 1-x, & x < 1 \\ (1-x)(2-x), & 1 \leq x \leq 2 \\ 3-x, & x > 2 \end{cases} \text{ Justify your answer.}$$

20. Let  $f(x), x \geq 0$ , be a non-negative continuous function, and

$$\text{let } F(x) = \int_0^x f(t) dt, x \geq 0. \text{ If for some } c > 0, f(x) \leq cF(x) \text{ for all}$$

$x \geq 0$ , then show that  $f(x) = 0$  for all  $x \geq 0$ . (2001 - 5 Marks)

21. Let  $\alpha \in R$ . Prove that a function  $f: R \rightarrow R$  is differentiable at  $\alpha$  if and only if there is a function  $g: R \rightarrow R$  which is continuous at  $\alpha$  and satisfies  $f(x) - f(\alpha) = g(x)(x - \alpha)$  for all  $x \in R$ . (2001 - 5 Marks)

22. Let  $f(x) = \begin{cases} x+a & \text{if } x < 0 \\ |x-1| & \text{if } x \geq 0, \end{cases}$  and (2002 - 5 Marks)

$$g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \geq 0, \end{cases} \text{ where } a \text{ and } b \text{ are}$$

non-negative real numbers. Determine the composite function  $g \circ f$ . If  $(g \circ f)(x)$  is continuous for all real  $x$ , determine the values of  $a$  and  $b$ . Further, for these values of  $a$  and  $b$ , is  $g \circ f$  differentiable at  $x = 0$ ? Justify your answer.

**Limits, Continuity and Differentiability**

23. If a function  $f : [-2a, 2a] \rightarrow R$  is an odd function such that  $f(x) = f(2a - x)$  for  $x \in [a, 2a]$  and the left hand derivative at  $x = a$  is 0 then find the left hand derivative at  $x = -a$ .

(2003 - 2 Marks)

24.  $f'(0) = \lim_{n \rightarrow \infty} n f\left(\frac{1}{n}\right)$  and  $f(0) = 0$ . Using this find

$$\lim_{n \rightarrow \infty} \left( (n+1) \frac{2}{\pi} \cos^{-1}\left(\frac{1}{n}\right) - n \right), \left| \cos^{-1} \frac{1}{n} \right| < \frac{\pi}{2}$$

(2004 - 2 Marks)

25. If  $|c| \leq \frac{1}{2}$  and  $f(x)$  is a differentiable function at  $x = 0$  given

$$\text{by } f(x) = \begin{cases} b \sin^{-1}\left(\frac{c+x}{2}\right), & -\frac{1}{2} < x < 0 \\ \frac{1}{2}, & x = 0 \\ \frac{e^{ax/2} - 1}{x}, & 0 < x < \frac{1}{2} \end{cases}$$

Find the value of 'a' and prove that  $64b^2 = 4 - c^2$

(2004 - 4 Marks)

26. If  $f(x - y) = f(x) \cdot g(y) - f(y) \cdot g(x)$  and

$$g(x - y) = g(x) \cdot g(y) - f(x) \cdot f(y) \text{ for all } x, y \in R.$$

If right hand derivative at  $x = 0$  exists for  $f(x)$ . Find derivative of  $g(x)$  at  $x = 0$

(2005 - 4 Marks)

**F Integer Value Correct Type**

**DIRECTIONS (Q. 1 and 2) :** Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

	p	q	r	s	t
A	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
B	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
C	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
D	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>

1. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book. (1992 - 2 Marks)

**Column I**

- (A)  $\sin(\pi [x])$
- (B)  $\sin(\pi(x-[x]))$

**Column II**

- (p) differentiable everywhere
- (q) nowhere differentiable
- (r) not differentiable at 1 and -1

2. In the following  $[x]$  denotes the greatest integer less than or equal to  $x$ .

Match the functions in **Column I** with the properties in **Column II** and indicate your answer by darkening the appropriate bubbles in the  $4 \times 4$  matrix given in the ORS. (2007 - 6 marks)

**Column I**

- (A)  $x|x|$
- (B)  $\sqrt{|x|}$
- (C)  $x + [x]$
- (D)  $|x-1| + |x+1|$

**Column II**

- (p) continuous in  $(-1, 1)$
- (q) differentiable in  $(-1, 1)$
- (r) strictly increasing in  $(-1, 1)$
- (s) not differentiable at least at one point in  $(-1, 1)$

**DIRECTIONS (Q. 3) :** Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

3. Let  $f_1 : R \rightarrow R$ ,  $f_2 : [0, \infty) \rightarrow R$ ,  $f_3 : R \rightarrow R$  and  $f_4 : R \rightarrow [0, \infty)$  be defined by  $f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \geq 0; \end{cases}$

$$f_2(x) = x^2; f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \geq 0; \end{cases} \text{ and } f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0. \end{cases} \quad (\text{JEE Adv. 2014})$$

## List-I

- P.  $f_4$  is  
 Q.  $f_3$  is  
 R.  $f_2 \circ f_1$  is  
 S.  $f_2$  is

	P	Q	R	S
(a)	3	1	4	2
(c)	3	1	2	4

## I Integer Value Correct Type

1. Let  $f: [1, \infty) \rightarrow [2, \infty)$  be a differentiable function such that

$$f(1) = 2. \text{ If } \int_1^x f(t) dt = 3xf(x) - x^3 \text{ for all } x \geq 1, \text{ then the}$$

value of  $f(2)$  is (2011)

2. The largest value of non-negative integer  $a$  for which

$$\lim_{x \rightarrow 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4} \text{ is (JEE Adv. 2014)}$$

3. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  be respectively given by  $f(x) = |x| + 1$  and  $g(x) = x^2 + 1$ . Define  $h: \mathbb{R} \rightarrow \mathbb{R}$  by

## List-II

1. Onto but not one-one  
 2. Neither continuous nor one-one  
 3. Differentiable but not one-one  
 4. Continuous and one-one

	P	Q	R	S
(b)	1	3	4	2
(d)	1	3	2	4

$$h(x) = \begin{cases} \max \{f(x), g(x)\} & \text{if } x \leq 0, \\ \min \{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which  $h(x)$  is not differentiable is (JEE Adv. 2014)

4. Let  $m$  and  $n$  be two positive integers greater than 1. If

$$\lim_{\alpha \rightarrow 0} \left( \frac{e^{\cos(\alpha^n)} - e}{\alpha^m} \right) = -\left(\frac{e}{2}\right) \text{ then the value of } \frac{m}{n} \text{ is}$$

(JEE Adv. 2015)

5. Let  $\alpha, \beta \in \mathbb{R}$  be such that  $\lim_{x \rightarrow 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$ . Then  $6(\alpha + \beta)$  equals. (JEE Adv. 2016)

## Section-B

## JEE Main / AIEEE

1.  $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2x}}$  is [2002]

- (a) 1 (b) -1  
 (c) zero (d) does not exist

2.  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + 5x + 3}{x^2 + x + 3} \right)^x$  [2002]

- (a)  $e^4$  (b)  $e^2$  (c)  $e^3$  (d) 1

3. Let  $f(x) = 4$  and  $f'(x) = 4$ . Then  $\lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x-2}$  is given by [2002]

- (a) 2 (b) -2 (c) -4 (d) 3

4.  $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}$  is [2002]

- (a)  $\frac{1}{p+1}$  (b)  $\frac{1}{1-p}$  (c)  $\frac{1}{p} - \frac{1}{p-1}$  (d)  $\frac{1}{p+2}$

5.  $\lim_{x \rightarrow 0} \frac{\log x^n - [x]}{[x]}$ ,  $n \in \mathbb{N}$ , ( $[x]$  denotes greatest integer less than or equal to  $x$ ) [2002]

- (a) has value -1 (b) has value 0  
 (c) has value 1 (d) does not exist

6. If  $f(1) = 1, f'(1) = 2$ , then  $\lim_{x \rightarrow 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$  is [2002]

- (a) 2 (b) 4 (c) 1 (d) 1/2

7.  $f$  is defined in  $[-5, 5]$  as [2002]

$$f(x) = x \text{ if } x \text{ is rational} \\ = -x \text{ if } x \text{ is irrational. Then}$$

- (a)  $f(x)$  is continuous at every  $x$ , except  $x = 0$   
 (b)  $f(x)$  is discontinuous at every  $x$ , except  $x = 0$   
 (c)  $f(x)$  is continuous everywhere  
 (d)  $f(x)$  is discontinuous everywhere

8.  $f(x)$  and  $g(x)$  are two differentiable functions on  $[0, 2]$  such that  $f''(x) - g''(x) = 0, f'(1) = 2g'(1) = 4f(2) = 3g(2) = 9$  then  $f(x) - g(x)$  at  $x = 3/2$  is [2002]

- (a) 0 (b) 2 (c) 10 (d) 5



## Limits, Continuity and Differentiability

9. If  $f(x+y) = f(x) \cdot f(y) \forall x, y$  and  $f(5) = 2, f'(0) = 3$ , then  $f'(5)$  is [2002]  
 (a) 0 (b) 1 (c) 6 (d) 2
10.  $\lim_{n \rightarrow \infty} \frac{1+2^4+3^4+\dots+n^4}{n^5} - \lim_{n \rightarrow \infty} \frac{1+2^3+3^3+\dots+n^3}{n^5}$  [2003]  
 (a)  $\frac{1}{5}$  (b)  $\frac{1}{30}$  (c) Zero (d)  $\frac{1}{4}$
11. If  $\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = k$ , the value of  $k$  is [2003]  
 (a)  $-\frac{2}{3}$  (b) 0 (c)  $-\frac{1}{3}$  (d)  $\frac{2}{3}$
12. The value of  $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sec^2 t dt}{x \sin x}$  is [2003]  
 (a) 0 (b) 3 (c) 2 (d) 1
13. Let  $f(a) = g(a) = k$  and their  $n$ th derivatives  $f^n(a), g^n(a)$  exist and are not equal for some  $n$ . Further if  $\lim_{x \rightarrow a} \frac{f(a)g(x) - f(x)g(a)}{g(x) - f(x)} = 4$ , then the value of  $k$  is [2003]  
 (a) 0 (b) 4 (c) 2 (d) 1
14.  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left[1 - \tan\left(\frac{x}{2}\right)\right][1 - \sin x]}{\left[1 + \tan\left(\frac{x}{2}\right)\right][\pi - 2x]^3}$  is [2003]  
 (a)  $\infty$  (b)  $\frac{1}{8}$  (c) 0 (d)  $\frac{1}{32}$
15. If  $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$  then  $f(x)$  is [2003]  
 (a) discontinuous every where  
 (b) continuous as well as differentiable for all  $x$   
 (c) continuous for all  $x$  but not differentiable at  $x = 0$   
 (d) neither differentiable nor continuous at  $x = 0$
16. If  $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$ , then the values of  $a$  and  $b$ , are [2004]  
 (a)  $a = 1$  and  $b = 2$  (b)  $a = 1, b \in \mathbf{R}$   
 (c)  $a \in \mathbf{R}, b = 2$  (d)  $a \in \mathbf{R}, b \in \mathbf{R}$
17. Let  $f(x) = \frac{1 - \tan x}{4x - \pi}, x \neq \frac{\pi}{4}, x \in \left[0, \frac{\pi}{2}\right]$ . If  $f(x)$  is continuous in  $\left[0, \frac{\pi}{2}\right]$ , then  $f\left(\frac{\pi}{4}\right)$  is [2004]  
 (a)  $-1$  (b)  $\frac{1}{2}$  (c)  $-\frac{1}{2}$  (d) 1
18.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right]$  equals [2005]  
 (a)  $\frac{1}{2} \sec 1$  (b)  $\frac{1}{2} \operatorname{cosec} 1$   
 (c)  $\tan 1$  (d)  $\frac{1}{2} \tan 1$
19. Let  $\alpha$  and  $\beta$  be the distinct roots of  $ax^2 + bx + c = 0$ , then  $\lim_{x \rightarrow \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$  is equal to [2005]  
 (a)  $\frac{a^2}{2}(\alpha - \beta)^2$  (b) 0  
 (c)  $-\frac{a^2}{2}(\alpha - \beta)^2$  (d)  $\frac{1}{2}(\alpha - \beta)^2$
20. Suppose  $f(x)$  is differentiable at  $x = 1$  and  $\lim_{h \rightarrow 0} \frac{1}{h} f(1+h) = 5$ , then  $f'(1)$  equals [2005]  
 (a) 3 (b) 4 (c) 5 (d) 6
21. Let  $f$  be differentiable for all  $x$ . If  $f(1) = -2$  and  $f'(x) \geq 2$  for  $x \in [1, 6]$ , then [2005]  
 (a)  $f(6) \geq 8$  (b)  $f(6) < 8$  (c)  $f(6) < 5$  (d)  $f(6) = 5$
22. If  $f$  is a real valued differentiable function satisfying  $|f(x) - f(y)| \leq (x - y)^2, x, y \in \mathbf{R}$  and  $f(0) = 0$ , then  $f(1)$  equals [2005]  
 (a)  $-1$  (b) 0 (c) 2 (d) 1
23. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a function defined by  $f(x) = \min\{x+1, |x|+1\}$ , Then which of the following is true?  
 (a)  $f(x)$  is differentiable everywhere [2007]  
 (b)  $f(x)$  is not differentiable at  $x = 0$   
 (c)  $f(x) \geq 1$  for all  $x \in \mathbf{R}$   
 (d)  $f(x)$  is not differentiable at  $x = 1$
24. The function  $f: \mathbf{R} \setminus \{0\} \rightarrow \mathbf{R}$  given by [2007]  

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1}$$
 can be made continuous at  $x = 0$  by defining  $f(0)$  as  
 (a) 0 (b) 1 (c) 2 (d)  $-1$

25. Let  $f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$  [2008]

Then which one of the following is true?

- (a)  $f$  is neither differentiable at  $x=0$  nor at  $x=1$   
 (b)  $f$  is differentiable at  $x=0$  and at  $x=1$   
 (c)  $f$  is differentiable at  $x=0$  but not at  $x=1$   
 (d)  $f$  is differentiable at  $x=1$  but not at  $x=0$

26. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a positive increasing function with

$\lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)} = 1$ . Then  $\lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} =$  [2010]

- (a)  $\frac{2}{3}$  (b)  $\frac{3}{2}$  (c) 3 (d) 1

27.  $\lim_{x \rightarrow 2} \left( \frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2} \right)$  [2011]

- (a) equals  $\sqrt{2}$  (b) equals  $-\sqrt{2}$   
 (c) equals  $\frac{1}{\sqrt{2}}$  (d) does not exist

28. The values of  $p$  and  $q$  for which the function [2011]

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$$

$x=0$  is continuous for all  $x$  in  $\mathbf{R}$ , are

- (a)  $p = \frac{5}{2}, q = \frac{1}{2}$  (b)  $p = -\frac{3}{2}, q = \frac{1}{2}$   
 (c)  $p = \frac{1}{2}, q = \frac{3}{2}$  (d)  $p = \frac{1}{2}, q = -\frac{3}{2}$

29. Let  $f: \mathbf{R} \rightarrow [0, \infty)$  be such that  $\lim_{x \rightarrow 5} f(x)$  exists and

$\lim_{x \rightarrow 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$ . Then  $\lim_{x \rightarrow 5} f(x)$  equals:

- (a) 0 (b) 1 (c) 2 (d) 3

30. If  $f: \mathbf{R} \rightarrow \mathbf{R}$  is a function defined by  $f(x) = [x]$

$\cos\left(\frac{2x-1}{2}\right)\pi$ , where  $[x]$  denotes the greatest integer function, then  $f$  is. [2012]

- (a) continuous for every real  $x$ .  
 (b) discontinuous only at  $x=0$   
 (c) discontinuous only at non-zero integral values of  $x$ .  
 (d) continuous only at  $x=0$ .

31. Consider the function,  $f(x) = |x-2| + |x-5|, x \in \mathbf{R}$ .

Statement-1:  $f'(4) = 0$

Statement-2:  $f$  is continuous in  $[2,5]$ , differentiable in  $(2,5)$  and  $f(2) = f(5)$ . [2012]

- (a) Statement-1 is false, Statement-2 is true.  
 (b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.  
 (c) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for Statement-1.  
 (d) Statement-1 is true, statement-2 is false.

32.  $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$  is equal to [JEE M 2013]

- (a)  $-\frac{1}{4}$  (b)  $\frac{1}{2}$  (c) 1 (d) 2

33.  $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$  is equal to: [JEE M 2014]

- (a)  $-\pi$  (b)  $\pi$  (c)  $\frac{\pi}{2}$  (d) 1

34.  $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$  is equal to: [JEE M 2015]

- (a) 2 (b)  $\frac{1}{2}$  (c) 4 (d) 3

35. If the function.

$$g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$$

is differentiable, then the value

of  $k+m$  is: [JEE M 2015]

- (a)  $\frac{10}{3}$  (b) 4 (c) 2 (d)  $\frac{16}{5}$

36. For  $x \in \mathbf{R}$ ,  $f(x) = |\log 2 - \sin x|$  and  $g(x) = f(f(x))$ , then:

[JEE M 2016]

- (a)  $g'(0) = -\cos(\log 2)$   
 (b)  $g$  is differentiable at  $x=0$  and  $g'(0) = -\sin(\log 2)$   
 (c)  $g$  is not differentiable at  $x=0$   
 (d)  $g'(0) = \cos(\log 2)$

37.  $\lim_{n \rightarrow \infty} \left( \frac{(n+1)(n+2)\dots 3n}{n^{2n}} \right)^{\frac{1}{n}}$  is equal to: [JEE M 2016]

- (a)  $\frac{9}{e^2}$  (b)  $3 \log 3 - 2$   
 (c)  $\frac{18}{e^4}$  (d)  $\frac{27}{e^2}$

38. Let  $p = \lim_{x \rightarrow 0^+} \left( 1 + \tan^2 \sqrt{x} \right)^{\frac{1}{2x}}$  then  $\log p$  is equal to:

[JEE M 2016]

- (a)  $\frac{1}{2}$  (b)  $\frac{1}{4}$   
 (c) 2 (d) 1



# Limits, Continuity and Differentiability

## Section-A : JEE Advanced/ IIT-JEE

- A**
1.  $\{0\}$       2.  $k=7$       3.  $f(x) = \sqrt{4-x^2}, -2 \leq x \leq 0 = -\sqrt{4-x^2}, 0 \leq x \leq 2$
4.  $\frac{2}{\pi}$       5. 1      6. -1      7. 4      8.  $\sqrt{2rh-h^2}, \frac{1}{128r}$       9.  $e^5$
10.  $R - \{0\}$       11.  $(-\infty, -1) \cup [0, \infty), I - \{0\}$  where  $I$  is the set of integer except  $n = -1$
12.  $e^2$       13. 10
- B**
1.  $F$
- C**
1. (c)      2. (d)      3. (a)      4. (d)      5. (c)      6. (b)      7. (b)
8. (d)      9. (a)      10. (b)      11. (c)      12. (b)      13. (d)      14. (d)
15. (c)      16. (c)      17. (b)      18. (a)      19. (d)      20. (d)      21. (d)
22. (c)      23. (c)      24. (d)      25. (d)      26. (c)      27. (a)      28. (b)
29. (c)      30. (a)      31. (a)      32. (c)      33. (d)      34. (b)      35. (b)
36. (b)
- D**
1. (a, b, d)      2. (b)      3. (b, d, e)      4. (a, b, d)      5. (a)      6. (a, b, c)
7. (b)      8. (d)      9. (b, c)      10. (b, c, d)      11. (a, b)      12. (a, c)
13. (a, c, d)      14. (d)      15. (a, d)      16. (a, c)      17. (b, c)      18. (a, b, c, d)
19. (b, d)      20. (b, d)      21. (a, c)      22. (a, d)      23. (a, d)      24. (a, b)
25. (b, c)
- E**
1.  $\frac{2}{3\sqrt{3}}$       2. 1      3.  $a^2 \cos a + 2a \sin a$       5.  $2 \ln 2$
6.  $g(x) = \begin{cases} 2+x, & 0 \leq x \leq 1 \\ 2-x, & 1 < x \leq 2 \\ 4-x, & 2 < x \leq 3 \end{cases}$  discontinuity at  $x = 1, 2$       7.  $f$  and  $f'$  are continuous and  $f''$  is discontinuous on  $[0, 2]$
8. cont. on  $(0, 2)$  and differentiable on  $(0, 2) - \{1\}$       9. not differentiable at  $x = 1$       11.  $f'(0) = 0$
12.  $a = \frac{\pi}{6}, b = -\frac{\pi}{12}$       13.  $x = 0, 1, 2, 3$       14.  $a = 8$       15.  $f(x) = e^{2x}$
16.  $e^2$       17.  $a = \frac{2}{3}, b = e^{2/3}$       18.  $f(2) = -1$
19.  $f$  is continuous and differentiable at all points except at  $x = 2$ .
22.  $g(f(x)) = \begin{cases} x+a+1. & \text{if } x < -a \\ (x+a-1)^2 + b & \text{if } a \leq x < 0 \\ x^2 + b & \text{if } 0 \leq x \leq 1 \\ (x-2)^2 + b & \text{if } x > 1 \end{cases}, a = 1, b = 0, \text{ gof is differentiable at } x = 0$       23. 0
24.  $\frac{\pi-2}{\pi}$       25. 1      26. 0
- F**
1. (A) - p, (B) - r      2. (A) - p, q, r; (B) - p, s; (C) - r, s; (D) - p, q      3. (d)
- I**
1. 6      2. 2      3. 3      4. 2      5. 7

**Section-B : JEE Main/ AIEEE**

1. (d)	2. (a)	3. (c)	4. (a)	5. (d)	6. (a)	7. (b)
8. (d)	9. (c)	10. (a)	11. (d)	12. (d)	13. (b)	14. (d)
15. (c)	16. (b)	17. (c)	18. (d)	19. (a)	20. (c)	21. (a)
22. (b)	23. (a)	24. (b)	25. (c)	26. (d)	27. (d)	28. (b)
29. (d)	30. (a)	31. (c)	32. (d)	33. (b)	34. (a)	35. (c)
36. (d)	37. (d)	38. (a)				

**Section-A JEE Advanced/ IIT-JEE**

**A. Fill in the Blanks**

1. Given  $f(x) = \begin{cases} (x-1)^2 \sin \frac{1}{x-1} - |x|, & x \neq 1 \\ -1, & x = 1 \end{cases}$   
 We know that  $|x|$  is not differentiable at  $x = 0$   
 $\therefore (x-1)^2 \sin \frac{1}{x-1} - |x|$  is not differentiable at  $x = 0$ .

At all other values of  $x$ ,  $f(x)$  is differentiable.  
 $\therefore$  The req. set of points is  $\{0\}$ .

2. It will be continuous at  $x = 2$  if

$$\lim_{x \rightarrow 2} f(x) = f(2) \Rightarrow \lim_{x \rightarrow 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2} = k$$

$$\Rightarrow k = \lim_{x \rightarrow 2} \frac{(x-2)^2(x+5)}{(x-2)^2} = \lim_{x \rightarrow 2} (x+5) = 7$$

$\therefore k = 7$

3.  $f(x) = \sqrt{4-x^2}, -2 \leq x \leq 0 = -\sqrt{4-x^2}, 0 \leq x \leq 2$

By choosing any arcs of circle  $x^2 - y^2 = 4$ , we can define a discontinuous function, one of which is

$$f(x) = \begin{cases} \sqrt{4-x^2}, & -2 \leq x \leq 0 \\ -\sqrt{4-x^2}, & 0 \leq x \leq 2 \end{cases}$$

**4. KEY CONCEPT**

**(L' Hospital rule)**

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if  $\frac{f(a)}{g(a)}$  is of the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  or  $0 \times \infty$ .

$$\lim_{x \rightarrow 1} (1-x) \tan \frac{\pi x}{2} \quad [\text{form } 0 \times \infty]$$

$$= \lim_{x \rightarrow 1} \frac{1-x}{\cot(\pi x/2)} \quad \left[ \text{Form } \frac{0}{0} \right]$$

$$= \lim_{x \rightarrow 1} \frac{-1}{\frac{-\pi}{2} \operatorname{cosec}^2 \left( \frac{\pi x}{2} \right)} \quad [\text{Applying L' Hospital's rule}]$$

$$= \frac{2}{\pi}$$

5. Given that,

$$f(x) = \sin x, x \neq \pi, n = 0, \pm 1, \pm 2, \dots = 2, \text{ otherwise}$$

$$\text{And } g(x) = x^2 + 1, x \neq 0, 2$$

$$= 4, x = 0 = 5, \quad x = 2$$

$$\text{Then } \lim_{x \rightarrow 0} g[f(x)] = \lim_{x \rightarrow 0} g(\sin x) \Rightarrow \lim_{x \rightarrow 0} (\sin^2 x + 1) = 1$$

6. 
$$\lim_{x \rightarrow -\infty} \left[ \frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{(1+|x|^3)} \right]$$

$$\text{Let } L = \lim_{x \rightarrow -\infty} \frac{x^3}{1+|x|^3} \left[ x \sin\left(\frac{1}{x}\right) + \frac{1}{x} \right]$$

$$= \lim_{x \rightarrow -\infty} \frac{x^3}{|x|^3} \left[ \frac{1}{1 + \frac{1}{|x|^2}} \right] \left[ x \sin\left(\frac{1}{x}\right) + \frac{1}{x} \right] \dots (1)$$

$$= \lim_{x \rightarrow -\infty} \frac{x^3}{|x|^3} \cdot 1 = \lim_{x \rightarrow -\infty} \frac{x^3}{-x^3} = -1$$

7. Given that  $f(9) = 9, f'(9) = 4$

Then,

$$\lim_{x \rightarrow 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3} = \lim_{x \rightarrow 9} \frac{(\sqrt{f(x)} - 3)(\sqrt{f(x)} + 3)}{(\sqrt{x} - 3)(\sqrt{x} + 3)}$$

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} + 3}{\sqrt{f(x)} + 3} = \lim_{x \rightarrow 9} \frac{f(x) - 9}{x - 9} \cdot \left[ \frac{3+3}{3+3} \right]$$

$$= \lim_{x \rightarrow 9} \frac{f(x) - f(9)}{x - 9} \cdot 1 = f'(9) = 4$$

8. In  $\Delta ABC, AB = AC$

$AD \perp BC$  (D is mid pt of BC)

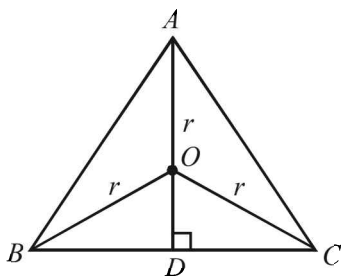
Let  $r =$  radius of circumcircle

$$\therefore OA = OB = OC = r$$

$$\text{Now } BD = \sqrt{BO^2 - OD^2} = \sqrt{r^2 - (h-r)^2}$$

$$= \sqrt{2rh - h^2}$$

$$\therefore BC = 2\sqrt{2rh - h^2}$$



∴ Area of  $\Delta ABC = \frac{1}{2} \times BC \times AD = h\sqrt{2rh - h^2}$

Also  $\lim_{h \rightarrow 0} \frac{A}{P^3} = \lim_{h \rightarrow 0} \frac{h\sqrt{2rh - h^2}}{8(\sqrt{2rh - h^2} + \sqrt{2rh})^3}$   
 $= \lim_{h \rightarrow 0} \frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}(\sqrt{2r-h} + \sqrt{2r})^3}$   
 $= \lim_{h \rightarrow 0} \frac{\sqrt{2r-h}}{8(2r-h + \sqrt{2r})^3}$   
 $= \frac{\sqrt{2r}}{8(\sqrt{2r} + \sqrt{2r})^3} = \frac{\sqrt{2r}}{8 \cdot 8 \cdot 2r \cdot \sqrt{2r}} = \frac{1}{128r}$

9.  $\lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1}\right)^{x+4} = \lim_{x \rightarrow \infty} \left\{ \left[1 + \frac{5}{x+1}\right]^{\frac{x+1}{5}} \right\}^{5\left(\frac{x+4}{x+1}\right)}$   
 [Using  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ ]

$\lim_{x \rightarrow \infty} 5\left(\frac{x+4}{x+1}\right) = e^{5 \lim_{x \rightarrow \infty} \left(\frac{1+4/x}{1+1/x}\right)} = e^5$

10. We have,

$f(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

$f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$

$f''(x) = \begin{cases} -2, & x < 0 \\ 2, & x \geq 0 \end{cases}$

Clearly  $f''(x)$  exists at every pt. except at  $x = 0$ . Thus  $f(x)$  is twice differentiable on  $R - \{0\}$ .

11. Thus function is not defined for those values of  $x$  for which  $[x+1] = 0$ . In other words it means that  $0 \leq x+1 < 1$  or  $-1 \leq x < 0$  .....(1)  
 Hence the function is defined outside the region given by (1).

Required domain is  $]-\infty, -1[ \cup ]0, \infty[$

Now, consider integral values of  $x$  say  $x = n$

R.H.L. =  $\lim_{h \rightarrow 0} [n+h] \sin \frac{\pi}{[n+1+h]} = n \sin \frac{\pi}{(n+1)}$

L.H.L. =  $\lim_{h \rightarrow 0} [n-h] \sin \frac{\pi}{[n+1-h]} = (n-1) \frac{\pi}{n}$

Clearly RHL  $\neq$  LHL. Hence the given function is not continuous for integral values of  $n$  ( $n \neq 0, -1$ ).

At  $x=0, f(0)=0,$

$\lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} [h] \sin \frac{\pi}{[h+1]} = 0$

The function is not defined for  $x < 0$ . Hence we cannot find  $\lim_{h \rightarrow 0} f(0-h)$ . Thus  $f(x)$  is continuous at  $x = 0$ . Hence the points of discontinuity are given by  $I - \{0\}$  where  $I$  is set of integers  $n$  except  $n = -1$

12. KEY CONCEPT

$\lim_{x \rightarrow 0} [f(x)]^{g(x)} = e^{\lim_{x \rightarrow 0} g(x) \log f(x)}$

$\lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = e^{\lim_{x \rightarrow 0} \frac{1}{x^2} \log \left[\frac{1+5x^2}{1+3x^2}\right]}$   
 $= e^{\lim_{x \rightarrow 0} \left[5 \cdot \frac{\log(1+5x^2)}{5x^2} - 3 \cdot \frac{\log(1+3x^2)}{3x^2}\right]}$   
 $= e^{5-3} = e^2$

13. Since  $f(x)$  is given continuous on the closed bounded interval  $[1, 3], f(x)$  is bounded and assumes all the values lying in the interval  $[m, M]$  where

$m = \min f(x)$  and  $M = \max f(x)$   
 $1 \leq x \leq 3 \Rightarrow f(1) \leq f(x) \leq (3)$

If  $m \geq M$ , then  $f(x)$  must assume all the irrational values lying in the  $[m, M]$ . But since  $f(x)$  takes only rational values, we must have  $m = M$  i.e.,  $f(x)$  must be a constant function. As  $f(2) = 10$ , we get

$f(x) = 10 \forall x \in [1, 3] \Rightarrow f(1.5) = 10$

B. True/False

1. Consider  $f(x) = \frac{|x-a|}{x-a}, g(x) = \frac{x-a}{|x-a|}$

then  $\lim_{x \rightarrow a} (f(x)g(x))$  exists but neither  $\lim_{x \rightarrow a} f(x)$  nor

$\lim_{x \rightarrow a} g(x)$  exists.

C. MCQs with ONE Correct Answer

1. (c)  $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$

$\Rightarrow \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \sqrt{\frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos^2 x}{x}}} = \sqrt{\frac{1-0}{1+0}} = 1$

2. (d)  $f(x) = \frac{\tan(\pi[x-\pi])}{1+[x]^2}$

By def.  $[x-\pi]$  is an integer whatever be the value of  $x$ . And so  $\pi[x-\pi]$  is an integral multiple of  $\pi$ .

Consequently  $\tan(\pi[x-\pi]) = 0, \forall x$ .

And since  $1 + [x]^2 \neq 0$  for any  $x$ , we conclude that  $f(x) = 0$ .

## Limits, Continuity and Differentiability

Thus  $f(x)$  is constant function and so, it is continuous and differentiable any no. of times, that is  $f'(x), f''(x), f'''(x), \dots$  all exist for every  $x$ , their value being 0 at every pt.  $x$ . Hence, out of all the alternatives only (d) is correct.

3. (a)  $f(x) = e^{-x}$  is one such function.

Here  $f(0) = 1, f'(0) = -1, f(x) > 0, \forall x$ .

$$\therefore f''(x) > 0 \quad \forall x$$

4. (d) 
$$\lim_{x \rightarrow 1} \frac{-\sqrt{25-x^2} - (-\sqrt{24})}{x-1}$$

$$= \lim_{x \rightarrow 1} \frac{\sqrt{24} - \sqrt{25-x^2}}{x-1} \times \frac{\sqrt{24} + \sqrt{25-x^2}}{\sqrt{24} + \sqrt{25-x^2}}$$

$$= \lim_{x \rightarrow 1} \frac{x^2 - 1}{(x-1)[\sqrt{24} + \sqrt{25-x^2}]}$$

$$= \lim_{x \rightarrow 1} \frac{x+1}{[\sqrt{24} + \sqrt{25-x^2}]} = \frac{2}{2\sqrt{24}} = \frac{1}{2\sqrt{6}}$$

5. (c) 
$$\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x-a}$$

$$= \lim_{h \rightarrow 0} \frac{g(a+h)f(a) - g(a)f(a+h)}{h} \quad [\text{For } x = a+h]$$

$$= \lim_{h \rightarrow 0} \frac{g(a+h)f(a) - g(a)f(a) + g(a)f(a) - g(a)f(a+h)}{h}$$

$$= \lim_{h \rightarrow 0} f(a) \left[ \frac{g(a+h) - g(a)}{h} \right] - \lim_{h \rightarrow 0} g(a) \left[ \frac{f(a+h) - f(a)}{h} \right]$$

$$= f(a)g'(a) - g(a)f'(a) = 2 \times 2 - (-1) \times 1 = 5$$

6. (b) For  $f(x)$  to be continuous at  $x=0$

$$f(0) = \lim_{x \rightarrow 0} f(x)$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1+ax) - \ln(1-bx)}{x}$$

$$\left[ \text{Using } \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \right]$$

$$= a + b$$

7. (b) 
$$\lim_{n \rightarrow \infty} \left( \frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{1-n^2} = \lim_{n \rightarrow \infty} \frac{\frac{\Sigma n}{2}}{1-n^2} = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2(1-n^2)}$$

$$= \lim_{n \rightarrow \infty} \frac{1+1/n}{2 \left[ \frac{1}{n^2} - 1 \right]} = -1/2$$

8. (d) The given function can be restated as

$$f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & \text{if } x \in (-\infty, 0) \cup [1, \infty) \\ 0, & \text{if } x \in [0, 1) \end{cases}$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \frac{\sin[-h]}{[-h]}$$

$$= \lim_{x \rightarrow 0^-} \frac{\sin(-1)}{(-1)} = \sin 1$$

$$\text{And } \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} 0 = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) \text{ as } \sin 1 \neq 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

9. (a) We have  $f: R \rightarrow R$ , a differentiable function and  $f(1) = 4$

**NOTE THIS STEP**

$$\lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} dt = \lim_{x \rightarrow 1} \left[ \frac{t^2}{x-1} \right]_4^{f(x)}$$

$$= \lim_{x \rightarrow 1} \frac{(f(x))^2 - 16}{x-1} = \lim_{x \rightarrow 1} \frac{f(x) - 4}{x-1} \cdot \lim_{x \rightarrow 1} (f(x) + 4)$$

$$= f'(1) \cdot (f(1) + 4) = 8f'(1) \quad [\text{Using } f(1) = 4]$$

10. (b) We have  $f(x) = [\tan^2 x]$

$\tan x$  is an increasing function for  $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$$\therefore \tan\left(-\frac{\pi}{4}\right) < \tan x < \tan\left(\frac{\pi}{4}\right) \quad \text{NOTE THIS STEP}$$

$$\Rightarrow -1 < \tan x < 1 \Rightarrow 0 < \tan^2 x < 1$$

$$\Rightarrow [\tan^2 x] = 0$$

$$\text{Hence, } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} [\tan^2 x] = 0$$

Also  $f(0) = 0$

$\therefore f(x)$  is continuous at  $x = 0$

11. (c) When  $x$  is not an integer, both the functions  $[x]$  and

$\cos\left(\frac{2x-1}{2}\right)\pi$  are continuous.

$\therefore f(x)$  is continuous on all non integral points.

For  $x = n \in I$

$$\lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} [x] \cos\left(\frac{2x-1}{2}\right)\pi$$

$$= (n-1) \cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$$\lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} [x] \cos\left(\frac{2x-1}{2}\right)\pi$$

$$= n \cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$$\text{Also } f(n) = n \cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$\therefore f$  is continuous at all integral pts as well.

Thus,  $f$  is continuous everywhere.

12. (b) We have  $= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{n\sqrt{1+(r/n)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}} dx \quad \left[ \because \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{a_n} f\left(\frac{r}{n}\right) = \int_0^a f(x) dx \right]$$

$$= \left[ \sqrt{1+x^2} \right]_0^2 = \sqrt{5} - 1$$

13. (d) We have  $f(x) = [x]^2 - [x^2]$   
At  $x=0$ ,

$$\begin{aligned} \text{L.H.L.} &= \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} [-h]^2 - [(-h)^2] \\ &= \lim_{h \rightarrow 0} f(-1)^2 - [h^2] = \lim_{h \rightarrow 0} 1 - 0 = 1 \end{aligned}$$

$$\begin{aligned} \text{R.H.L.} &= \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} [h]^2 - [h^2] \\ &= \lim_{h \rightarrow 0} 0 - 0 = 0 \end{aligned}$$

$\therefore$  L.H.L.  $\neq$  R.H.L.

$\therefore f(x)$  is not continuous at  $x=0$ .

At  $x=1$

$$\begin{aligned} \text{L.H.L.} &= \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} [1-h]^2 - [(1-h)^2] \\ &= \lim_{h \rightarrow 0} 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{R.H.L.} &= \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} [1+h]^2 - [(1+h)^2] \\ &= \lim_{h \rightarrow 0} 1 - 1 = 0 \end{aligned}$$

$$f(1) = [1]^2 - [1^2] = 1 - 1 = 0$$

$\therefore$  L.H.L. = R.H.L. =  $f(1)$

$\therefore f(x)$  is continuous at  $x=1$ .

Clearly at other integral pts  $f(x)$  is not continuous.

14. (d) We have  $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$$\text{Also, } |x^2 - 3x + 2| = |(x-1)(x-2)|$$

$$= \begin{cases} (1-x)(2-x) & \text{if } x < 1 \\ (x-1)(2-x) & \text{if } 1 \leq x < 2 \\ (x-1)(x-2) & \text{if } x \geq 2 \end{cases}$$

As  $\cos(-\theta) = \cos \theta \Rightarrow \cos |x| = \cos x$

$\therefore$  Given function can be written as

$$f(x) = \begin{cases} (x^2 - 1)(x-1)(x-2) + \cos x & \text{if } x \leq 1 \\ -(x^2 - 1)(x-1)(x-2) + \cos x & \text{if } 1 \leq x < 2 \\ (x^2 - 1)(x-1)(x-2) + \cos x & \text{if } x \geq 2 \end{cases}$$

This function is differentiable at all points except possibly at  $x=1$  and  $x=2$ .

$$\begin{aligned} \text{Lf}'(1) &= \left\{ \frac{d}{dx} [(x^2 - 1)(x-1)(x-2) + \cos x] \right\}_{x=1} \\ &= -\sin 1 \end{aligned}$$

$$\begin{aligned} \text{Rf}'(1) &= \left\{ \frac{d}{dx} [-(x^2 - 1)(x-1)(x-2) + \cos x] \right\}_{x=1} \\ &= -\sin 1 \end{aligned}$$

$\therefore$  Lf'(1) = Rf'(1)

$\therefore f$  is differentiable at  $x=1$ .

$$\begin{aligned} \text{Lf}'(2) &= \left\{ \frac{d}{dx} [-(x^2 - 1)(x-1)(x-2) + \cos x] \right\}_{x=2} \\ &= -3 - \sin 2 \end{aligned}$$

$$\begin{aligned} \text{Rf}'(2) &= \left\{ \frac{d}{dx} [(x^2 - 1)(x-1)(x-2) + \cos x] \right\}_{x=2} \\ &= 3 - \sin 2 \end{aligned}$$

$$\text{Lf}'(2) \neq \text{Rf}'(2)$$

$\therefore f$  is not differentiable at  $x=2$ .

$$\begin{aligned} 15. (c) \quad \lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2} \\ &= \lim_{x \rightarrow 0} \frac{x \left\{ 2x + \frac{8x^3}{3} + \frac{64x^5}{15} + \dots \right\} - 2x \left\{ x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \right\}}{4 \sin^4 x} \\ &= \lim_{x \rightarrow 0} \frac{x^4 \left\{ \frac{8}{3} - \frac{2}{3} + \text{terms containing higher positive powers of } x \right\}}{4 \sin^4 x} \\ &= \lim_{x \rightarrow 0} \frac{1}{4} \cdot 2 = \frac{1}{2} \end{aligned}$$

16. (c) For  $x \in R$ ,

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \frac{x-3}{x+2} \right)^x &= \lim_{x \rightarrow \infty} \left\{ \left[ 1 - \frac{5}{x+2} \right]^{\frac{-(x+2)}{5}} \right\}^{\frac{-5x}{x+2}} \\ &= e^{\lim_{x \rightarrow \infty} -\frac{5}{1+\frac{2}{x}}} = e^{-5} \end{aligned}$$

$$\begin{aligned} 17. (b) \quad \lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2} &= \lim_{x \rightarrow 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{(\pi \sin^2 x)}{x^2} = \pi \end{aligned}$$

$$\begin{aligned} 18. (a) \quad \text{At } LHD &= \lim_{h \rightarrow 0} \frac{f(k) - f(k-h)}{h} \quad (k = \text{integer}) \\ &= \lim_{h \rightarrow 0} \frac{[k] \sin k\pi - [k-h] \sin(k-h)\pi}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(k-1) \sin(k-h)\pi}{h} \quad [\because \sin k\pi = 0] \\ &= \lim_{h \rightarrow 0} \frac{-(k-1) \sin(k\pi - h\pi)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(k-1)(-1)^{k-1} \sin h\pi}{h\pi} \times \pi = \pi(k-1)(-1)^k \end{aligned}$$

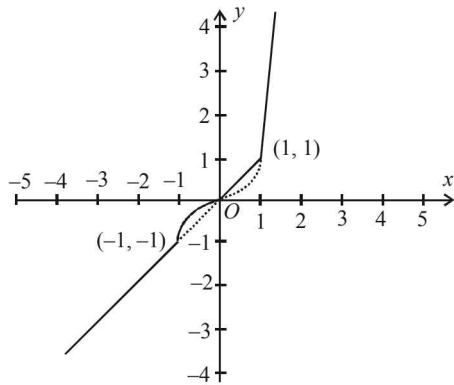
19. (d)  $f(x) = \max\{x, x^3\}$

$$= \begin{cases} x & ; & x < -1 \\ x^3 & ; & -1 \leq x \leq 0 \\ x & ; & 0 \leq x \leq 1 \\ x^3 & ; & x \geq 1 \end{cases}$$

#### KEY CONCEPT

A continuous function  $f(x)$  is not differentiable at  $x=a$  if graphically it takes a sharp turn at  $x=a$ .

Graph of  $f(x) = \max\{x, x^3\}$  is as shown with solid lines.



From graph of  $f(x)$  at  $x = -1, 0, 1$ , we have sharp turns.

$\therefore f(x)$  is not differentiable at  $x = -1, 0, 1$ .

20. (d) Let us test each of four options.

$$(a) f(x) = \cos |x| + |x| = \begin{cases} \cos x - x, & x < 0 \\ \cos x + x, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -\sin x - 1, & x < 0 \\ -\sin x + 1, & x \geq 0 \end{cases}$$

At  $x = 0$ ,  $LD = -1$ ,  $RD = 1$

$\therefore$  Not differentiable

$$(b) f(x) = \cos |x| - |x| = \begin{cases} \cos x + x, & x < 0 \\ \cos x - x, & x \geq 0 \end{cases}$$

$\therefore$  Not differentiable at  $x = 0$

$$(c) f(x) = \sin |x| + |x| = \begin{cases} -\sin x - x, & x < 0 \\ \sin x - x, & x \geq 0 \end{cases}$$

$\therefore$  Not differentiable at  $x = 0$

$$(d) f(x) = \sin |x| - |x| = \begin{cases} -\sin x + x, & x < 0 \\ \sin x - x, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -\cos x + 1, & x < 0 \\ \cos x - 1, & x \geq 0 \end{cases}$$

At  $x = 0$ ,  $LD = 0$ ,  $RD = 0$

$\therefore f$  is differentiable at  $x = 0$ .

21. (d) The given function is

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \leq 1 \\ \frac{1}{2}(|x| - 1) & \text{if } |x| > 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{2}(-x - 1) & \text{if } x < -1 \\ \tan^{-1} x & \text{if } -1 \leq x \leq 1 \\ \frac{1}{2}(x - 1) & \text{if } x > 1 \end{cases}$$

Clearly L.H.L. at  $(x = -1) = \lim_{h \rightarrow 0} f(-1 - h) = 0$

R.H.L. at  $(x = -1) = \lim_{h \rightarrow 0} f(-1 + h)$

$$= \lim_{h \rightarrow 0} \tan^{-1}(-1 + h) = 3\pi/4$$

$\therefore$  L.H.L.  $\neq$  R.H.L. at  $x = -1$

$\therefore f(x)$  is discontinuous at  $x = -1$

Also we can prove in the same way, that  $f(x)$  is discontinuous at  $x = 1$

$\therefore f'(x)$  can not be found for  $x = \pm 1$  or domain of

$$f'(x) = R - \{-1, 1\}$$

22. (c) Given that,

$$\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n} = \text{finite non zero number}$$

$$= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)(e^x - \cos x)}{x^n(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \left( \frac{\sin^2 x}{x^2} \right) \cdot \left( \frac{e^x - \cos x}{x^{n-2}} \right) \cdot \left( \frac{1}{1 + \cos x} \right)$$

$$= \lim_{x \rightarrow 0} 1^2 \cdot \frac{e^x - \cos x}{x^{n-2}} \cdot \frac{1}{2}$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{e^x - \sin x}{(x-2)x^{n-3}} \quad [\text{using L' Hospital's rule}]$$

For this limit to be finite,  $n - 3 = 0 \Rightarrow n = 3$

23. (c) Given that  $f: R \rightarrow R$  such that

$f(1) = 3$  and  $f'(1) = 6$

$$\text{Then } \lim_{x \rightarrow 0} \left[ \frac{f(1+x)}{f(1)} \right]^{1/x}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{x} [\log f(1+x) - \log f(1)]}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{f(1+x)} f'(1+x)}$$

[Using L' Hospital rule]

$$= e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$$

24. (d) We are given that

$$\lim_{x \rightarrow 0} \frac{[(a-n)nx - \tan x] \sin nx}{x^2} = 0$$

where  $n$  is non zero real number

$$\Rightarrow \lim_{x \rightarrow 0} n \cdot \frac{\sin nx}{nx} \left[ \left\{ (a-n)n - \frac{\tan x}{x} \right\} \right] = 0$$

$$\Rightarrow 1 \cdot n [(a-n)n - 1] = 0 \Rightarrow a = \frac{1}{n} + n$$

25. (d) Let  $L = \lim_{h \rightarrow 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)}$   $\left[ \frac{0}{0} \text{ form} \right]$

$\therefore$  Applying L Hospital's rule, we get

$$L = \lim_{h \rightarrow 0} \frac{f'(2h+2+h^2) \cdot (2+2h)}{f'(h-h^2+1) \cdot (1-2h)}$$

$$= \frac{f'(2) \cdot 2}{f'(1) \cdot 1} = \frac{6 \times 2}{4 \times 1} = 3$$

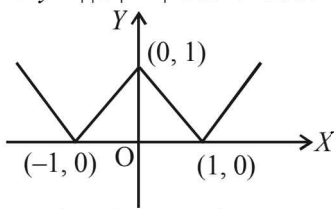
26. (c) Let  $L = \lim_{x \rightarrow 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$   $\left[ \because f'(a) > 0, f \text{ being strictly increasing} \right]$

Using L.H. Rule, we get

$$L = \lim_{x \rightarrow 0} \frac{f'(x^2) \cdot 2x - f'(x)}{f'(x)} = \lim_{x \rightarrow 0} \frac{f'(x^2) \cdot 2x}{f'(x)} - 1 = 0 - 1 = -1$$



27. (a) Graph of  $y = ||x| - 1|$  is as follows :



The graph has sharp turnings at  $x = -1, 0$  and  $1$ ; and hence not differentiable at  $x = -1, 0, 1$ .

28. (b) Given that  $f(x)$  is a continuous and differentiable function and  $f\left(\frac{1}{x}\right) = 0, x = n, n \in I$

$$\therefore f(0^+) = f\left(\frac{1}{\infty}\right) = 0$$

Since R.H.L. = 0,

$\therefore f(0) = 0$  for  $f(x)$  to be continuous.

$$\text{Also } f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \rightarrow 0} \frac{f(h)}{h} = 0$$

= 0 [Using  $f(0) = 0$  and  $f(0^+) = 0$ ]

Hence  $f(0) = 0, f'(0) = 0$

29. (c)  $\lim_{x \rightarrow 0} [(\sin x)^{1/x} + (1/x)^{\sin x}]$

$$\begin{aligned} &= \lim_{x \rightarrow 0} (\sin x)^{1/x} + \lim_{x \rightarrow 0} \left(\frac{1}{x}\right)^{\sin x} \\ &= 0 + e^{\lim_{x \rightarrow 0} \sin x \log\left(\frac{1}{x}\right)} \\ & \quad [\because |\sin x| < 1 \text{ when } x \rightarrow 0] \\ &= e^{\lim_{x \rightarrow 0} \frac{-\log x}{\operatorname{cosec} x}} = e^{\lim_{x \rightarrow 0} \frac{-1/x}{-\operatorname{cosec} x \cot x}} \\ & \quad \text{[Using L' Hospital rule]} \\ &= e^{\lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \tan x} = e^0 = 1 \end{aligned}$$

30. (a) Given that  $f(x)$  is differentiable on  $(0, \infty)$  with

$$\begin{aligned} f(1) = 1 \text{ and } \lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1 \text{ for each } x > 0 \\ \Rightarrow \lim_{t \rightarrow x} \frac{2t f(x) - x^2 f'(t)}{1} = 1 \text{ [Using L' Hospital rule]} \\ \Rightarrow 2x f(x) - x^2 f'(x) = 1 \Rightarrow f'(x) - \frac{2}{x} f(x) = -\frac{1}{x^2} \end{aligned}$$

[Linear differential equation]

Integrating factor  
**NOTE THIS STEP**

$$\begin{aligned} e^{\int -\frac{2}{x} dx} &= e^{-2 \log x} = e^{\log 1/x^2} = \frac{1}{x^2} \\ \therefore \text{Solution is } f(x) \times \frac{1}{x^2} &= \int \left(-\frac{1}{x^2}\right) \times \frac{1}{x^2} dx \\ \Rightarrow \frac{f(x)}{x^2} &= \frac{1}{3x^3} + C \Rightarrow f(x) = Cx^2 + \frac{1}{3x} \\ \text{Also } f(1) &= 1 \\ \Rightarrow 1 &= C + \frac{1}{3} \Rightarrow C = 2/3 \therefore f(x) = \frac{2}{3}x^2 + \frac{1}{3x} \end{aligned}$$

31. (a) **KEY CONCEPT**

$$\begin{aligned} \frac{d}{dx} \left[ \int_{g(x)}^{h(x)} f(t) dt \right] &= f(h(x))h'(x) - f(g(x))g'(x) \\ &= \frac{f(2) \times 2 \times 2 \times 1}{2 \times \frac{\pi}{4}} = \frac{8}{\pi} f(2) \end{aligned}$$

$$\text{Let } L = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}} \quad \left[ \frac{0}{0} \text{ form} \right]$$

On applying L' Hospital's rule, we get

$$\begin{aligned} L &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{d}{dx} \left[ \int_2^{\sec^2 x} f(t) dt \right]}{\frac{d}{dx} \left( x^2 - \frac{\pi^2}{16} \right)} \\ L &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{f(\sec^2 x) \cdot 2 \sec^2 x \tan x}{2x} \end{aligned}$$

32. (c) As per question,  
 $p =$  left hand derivative of  $|x-1|$  at  $x = 1 \Rightarrow p = -1$   
Also  $\lim_{x \rightarrow 1^+} g(x) = p$

$$\text{Where } g(x) = \frac{(x-1)^n}{\log \cos^m(x-1)}, 0 < x < 2,$$

$m, n$  are integers,  $m \neq 0, n > 0$

$\therefore$  we get,

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{(x-1)^n}{\log \cos^m(x-1)} &= -1 \Rightarrow \lim_{h \rightarrow 0} \frac{h^n}{\log \cos^m h} = -1 \\ \Rightarrow \lim_{h \rightarrow 0} \frac{h^n}{m(\log \cos h)} &= -1 \text{ [Using L' Hospital's rule]} \\ \Rightarrow \lim_{h \rightarrow 0} \frac{n h^{n-1} \cos h}{m(-\sin h)} &= -1 \text{ [Using L' Hospital's rule]} \\ \Rightarrow \lim_{h \rightarrow 0} \frac{n h^{n-2} \cosh}{m \left(\frac{\sinh}{h}\right)} &= 1 \Rightarrow n = 2 \text{ and } m = 2 \end{aligned}$$

33. (d)  $\lim_{x \rightarrow 0} [1 + x \ln(1 + b^2)]^{\frac{1}{x}} = 2b \sin^2 \theta$

$$\begin{aligned} &\Rightarrow e^{\lim_{x \rightarrow 0} \frac{1}{x} \ln[1 + x \ln(1 + b^2)]} = 2b \sin^2 \theta \\ &\Rightarrow e^{\lim_{x \rightarrow 0} \frac{\ln[1 + x \ln(1 + b^2)]}{x \ln(1 + b^2)} \times \ln(1 + b^2)} = 2b \sin^2 \theta \\ &\Rightarrow e^{\ln(1 + b^2)} = 2b \sin^2 \theta \\ &\Rightarrow 1 + b^2 = 2b \sin^2 \theta \Rightarrow 2 \sin^2 \theta = b + \frac{1}{b} \end{aligned}$$

We know that  $2 \sin^2 \theta \leq 2$  and  $b + \frac{1}{b} \geq 2$  for  $b > 0$

## Limits, Continuity and Differentiability

$$\therefore 2 \sin^2 \theta = b + \frac{1}{b} = 2 \Rightarrow \sin^2 \theta = 1$$

$$\text{As } \theta \in (-\pi, \pi], \therefore \theta = \pm \frac{\pi}{2}$$

$$\begin{aligned} 34. \quad (b) \quad \text{Given: } \lim_{x \rightarrow \infty} \left( \frac{x^2 + x + 1}{x + 1} - ax - b \right) &= 4 \\ \Rightarrow \lim_{x \rightarrow \infty} \frac{x^2 + x + 1 - ax^2 - ax - bx - b}{x + 1} &= 4 \\ \Rightarrow \lim_{x \rightarrow \infty} \frac{(1-a)x^2 + (1-a-b)x + (1-b)}{x + 1} &= 4 \end{aligned}$$

For this limit to be finite  $1-a=0 \Rightarrow a=1$   
then given limit reduces to

$$\lim_{x \rightarrow \infty} \frac{-bx + (1-b)}{x + 1} = 4 \Rightarrow \lim_{x \rightarrow \infty} \frac{-b + \frac{(1-b)}{x}}{1 + \frac{1}{x}} = 4$$

$$\Rightarrow -b = 4 \text{ or } b = -4$$

Hence  $a = 1, b = -4$ .

$$\begin{aligned} 35. \quad (b) \quad \text{We have } f'(0^+) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 \left| \cos \frac{\pi}{h} \right|}{h} = \lim_{h \rightarrow 0} h \left| \cos \frac{\pi}{h} \right| \\ &= 0 \times \text{some finite value} = 0 \end{aligned}$$

$$\begin{aligned} \text{Also, } f'(0^-) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{h^2 \left| \cos \frac{\pi}{-h} \right|}{-h} \\ &= \lim_{h \rightarrow 0} -h \left| \cos \frac{\pi}{h} \right| = 0 \times \text{some finite value} = 0 \end{aligned}$$

$\therefore f'(0^+) = f'(0^-) \Rightarrow f$  is differentiable at  $x=0$

$$\begin{aligned} \text{Now } f'(2^+) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2 \left| \cos \frac{\pi}{2+h} \right| - 4 \left| \cos \frac{\pi}{2} \right|}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2 \left( \cos \frac{\pi}{2+h} \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2}{h} \sin \left( \frac{\pi}{2} - \frac{\pi}{2+h} \right) \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2}{h} \sin \left( \frac{\pi h}{2(2+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2}{h} \times \frac{\sin \left( \frac{\pi h}{2(2+h)} \right)}{\left( \frac{\pi h}{2(2+h)} \right)} \times \frac{\pi h}{2(2+h)} = \pi \end{aligned}$$

$$\begin{aligned} \text{Also } f'(2^-) &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 \left| \cos \left( \frac{\pi}{2-h} \right) \right| - 4 \left| \cos \frac{\pi}{2} \right|}{-h} \\ &= \lim_{h \rightarrow 0} \frac{-(2-h)^2 \cos \left( \frac{\pi}{2-h} \right)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 \sin \left( \frac{\pi}{2} - \frac{\pi}{2-h} \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2}{h} \times \frac{\sin \left( \frac{-\pi h}{2(2-h)} \right)}{\left( \frac{-\pi h}{2(2-h)} \right)} \times \left( \frac{-\pi h}{2(2-h)} \right) = -\pi \end{aligned}$$

As  $f'(2^+) \neq f'(2^-) \Rightarrow f$  is not differentiable at  $x=2$ .

36. (b) The given equation is

$$(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$$

Let  $a+1=y$ , then equation reduces to

$$(y^{1/3} - 1)x^2 + (y^{1/2} - 1)x + (y^{1/6} - 1) = 0$$

Dividing both sides by  $y-1$ , we get

$$\left( \frac{y^{1/3} - 1}{y-1} \right) x^2 + \left( \frac{y^{1/2} - 1}{y-1} \right) x + \left( \frac{y^{1/6} - 1}{y-1} \right) = 0$$

Taking limit as  $y \rightarrow 1$  i.e.  $a \rightarrow 0$  on both sides we get

$$\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{6} = 0 \Rightarrow 2x^2 + 3x + 1 = 0$$

$$\Rightarrow x = -1, -\frac{1}{2} \text{ (roots of the equation)}$$

$$\text{Thus } \lim_{a \rightarrow 0^+} \alpha(a) = -1, \lim_{a \rightarrow 0^+} \beta(a) = -\frac{1}{2}$$

## D. MCQs with ONE or MORE THAN ONE Correct

1. (a, b, d) Given that  $x + |y| = 2y$

If  $y < 0$  then  $x - y = 2y$

$$\Rightarrow y = x/3 \Rightarrow x < 0$$

If  $y = 0$  then  $x = 0$ . If  $y > 0$  then  $x + y = 2y$

$$\Rightarrow y = x \Rightarrow x > 0$$

Thus we can define  $f(x) = y = \begin{cases} x/3, & x < 0 \\ x, & x \geq 0 \end{cases}$

**Continuity at  $x=0$**

$$LL = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} (-h/3) = 0$$

$$RL = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} h = 0$$

$$f(0) = 0$$

As  $LL = RL = f(0)$

$\therefore f(x)$  is continuous at  $x=0$

**Differentiability at  $x=0$**

$$Lf' = 1/3; Rf' = 1$$

As  $Lf' \neq Rf' \Rightarrow f(x)$  is not differentiable at  $x=0$

But for  $x < 0$ ,  $\frac{dy}{dx} = \frac{1}{3}$ .

2. (b) We have  $f(x) = x(\sqrt{x} - \sqrt{x+1})$

Let us check differentiability of  $f(x)$  at  $x=0$

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{(0-h)[\sqrt{0-h} - \sqrt{0-h+1}] - 0}{-h}$$

$$= \lim_{h \rightarrow 0} [\sqrt{-h} - \sqrt{-h+1}] = 0 - \sqrt{1} = -1$$

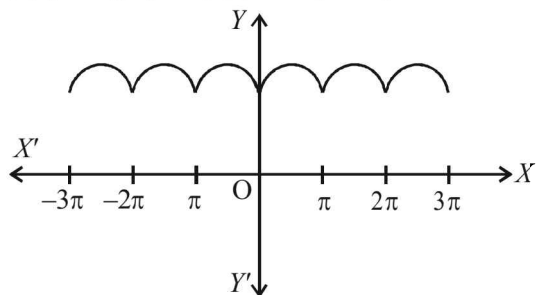
$$Rf'(0) = \lim_{h \rightarrow 0} \frac{(0+h)[\sqrt{0+h} - \sqrt{0+h+1}] - 0}{h}$$

$$= \lim_{h \rightarrow 0} \sqrt{h} - \sqrt{h+1} = -1$$

Since  $Lf'(0) = Rf'(0)$

$\therefore f$  is differentiable at  $x=0$ .

3. (b,d,e) The graph of  $f(x) = 1 + |\sin x|$  is as shown in the fig.



From graph it is clear that function is continuous every where but not differentiable at integral multiples of  $\pi$  ( $\because$  at these points curve has sharp turnings)

4. (a,b,d) We have, for  $-1 \leq x \leq 1 \Rightarrow 0 \leq x \sin \pi x \leq 1/2$

$$\therefore f(x) = [x \sin \pi x] = 0$$

Also  $x \sin \pi x$  becomes negative and numerically less than 1 when  $x$  is slightly greater than 1 and so by definition of  $[x]$ ,

$$f(x) = [x \sin \pi x] = -1 \text{ when } 1 < x < 1 + h$$

Thus  $f(x)$  is constant and equal to 0 in the closed interval  $[-1, 1]$  and so  $f(x)$  is continuous and differentiable in the open interval  $(-1, 1)$ .

At  $x=1$ ,  $f(x)$  is clearly discontinuous, since  $f(1-0) = 0$  and  $f(1+0) = -1$  and  $f(x)$  is non-differentiable at  $x=1$ .

5. (a) The given function is,

$$f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$$

$$\text{For } x < 0, f'(x) = \frac{1(1-x) - (-1)x}{(1-x)^2} = \frac{1}{(1-x)^2},$$

$$\text{For } x > 0, f'(x) = \frac{1(1+x) - 1(x)}{(1+x)^2} = \frac{1}{(1+x)^2},$$

For  $x=0$ ,

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{1+h} - 0}{-h} = 1$$

$$Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h}{1+h} - 0}{h} = 1$$

$$Lf'(0) = Rf'(0)$$

$\Rightarrow f$  is differentiable at  $x=0$

Hence  $f$  is differentiable in  $(-\infty, \infty)$ .

$$6. \text{ (a,b,c) } f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$$

$$= \begin{cases} \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \\ 3-x, & 1 \leq x < 3 \\ x-3, & x \geq 3 \end{cases}$$

$$Lf'(1) = \frac{2x}{4} - \frac{3}{2} = -1$$

$$Rf'(1) = -1. \text{ Thus } Lf'(1) = Rf'(1)$$

$\therefore f$  is differentiable at  $x=1$  and hence continuous at  $x=1$ .

$$\text{Again, } Lf'(3) = -1 \text{ and } Rf'(3) = 1$$

$$\Rightarrow Lf'(3) \neq Rf'(3)$$

$\Rightarrow f$  is not differentiable at  $x=3$

Let us now check the continuity at  $x=3$

$$\text{L.H.L.} = \lim_{h \rightarrow 0} f(3-h) = \lim_{h \rightarrow 0} [3 - (3-h)] = 0$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} f(3+h) = \lim_{h \rightarrow 0} [3 + h - 3] = 0$$

$$f(3) = 0$$

$\therefore f$  is continuous at  $x=3$

7. (b) We have,  $f(x) = \frac{x}{2} - 1$

$$\therefore [f(x)] = \left[ \frac{x}{2} - 1 \right] = -1, 0 \leq x < 2$$

$$\tan [f(x)] = \tan(-1), 0 \leq x < 2 = 0, 2 \leq x \leq \pi$$

$\therefore$  The function  $\tan [f(x)]$  is discontinuous at  $x=2$ .

$$\text{Also the function } \frac{1}{f(x)} = \frac{1}{\frac{x}{2} - 1} = \frac{2}{x-2} \text{ is}$$

discontinuous at  $x=2$ .

Thus both the given functions  $\tan [f(x)]$  as well as

$\frac{1}{f(x)}$  are discontinuous on the interval  $[0, \pi]$ .

Also  $f^{-1}(x) = y$

$$\Rightarrow x = f(y) = \frac{y}{2} - 1 \Leftrightarrow y = 2(x+1)$$

$\therefore f^{-1}(x) = 2(x+1)$  is continuous on  $[0, \pi]$

$$8. \text{ (d) } \lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2} \cdot 2 \sin^2 x}}{x} = \lim_{x \rightarrow 0} \frac{|\sin x|}{x}$$

Limits, Continuity and Differentiability

$$\therefore \text{L.H.L.} = \lim_{h \rightarrow 0} \frac{|\sin(0-h)|}{0-h} = \lim_{h \rightarrow 0} \frac{-\sin h}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin h}{h} = -1$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} \frac{|\sin(0+h)|}{0+h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

As L.H.L.  $\neq$  R.H.L.

$\therefore$  The given limit does not exist.

9. (b,c) On  $(0, \pi)$

(a)  $\tan x = f(x)$

We know that  $\tan x$  is discontinuous at  $x = \pi/2$

(b)  $f(x) = \int_0^x t \sin\left(\frac{1}{t}\right) dt$

NOTE THIS STEP

$\Rightarrow f'(x) = x \sin\left(\frac{1}{x}\right)$  which exists on  $(0, \pi)$

$\therefore f(x)$ , being differentiable, is continuous on  $(0, \pi)$ .

(c)  $f(x) = \begin{cases} 1 & , 0 < x \leq 3\pi/4 \\ 2 \sin \frac{2x}{9} & , 3\pi/4 < x < \pi \end{cases}$

Clearly  $f(x)$  is continuous on  $(0, \pi)$  except possibly at  $x = \frac{3\pi}{4}$ , where,

L.H.L. =  $\lim_{h \rightarrow 0} f\left(\frac{3\pi}{4} - h\right) = \lim_{x \rightarrow 0} 1 = 1$

R.H.L. =  $\lim_{h \rightarrow 0} f\left(\frac{3\pi}{4} + h\right) = \lim_{x \rightarrow 0} 2 \sin \frac{2}{9}\left(\frac{3\pi}{4} + h\right)$

=  $\lim_{h \rightarrow 0} 2 \sin\left(\frac{\pi}{6} + \frac{2h}{9}\right) = 2 \sin \frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$

Also  $f\left(\frac{3\pi}{4}\right) = 1$

As L.H.L. = R.H.L. =  $f\left(\frac{3\pi}{4}\right)$

$\therefore f(x)$  is continuous on  $(0, \pi)$

(d)  $f(x) = \begin{cases} x \sin x & , 0 < x \leq \pi/2 \\ \frac{\pi}{2} \sin(\pi + x) & , \frac{\pi}{2} < x < \pi \end{cases}$

Here  $f(x)$  will be continuous on  $(0, \pi)$  if it is continuous at  $x = \pi/2$ .

At  $x = \pi/2$ ,

L.H.L. =  $\lim_{h \rightarrow 0} f\left(\frac{\pi}{2} - h\right)$

=  $\lim_{h \rightarrow 0} \left(\frac{\pi}{2} - h\right) \sin\left(\frac{\pi}{2} - h\right) = \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2}$

R.H.L. =  $\lim_{h \rightarrow 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h \rightarrow 0} \frac{\pi}{2} \sin\left(\pi + \frac{\pi}{2} + h\right)$

=  $\frac{\pi}{2} \sin\left(\pi + \frac{\pi}{2}\right) = \frac{-\pi}{2} \sin \frac{\pi}{2} = -\frac{\pi}{2}$

As L.H.L.  $\neq$  R.H.L.

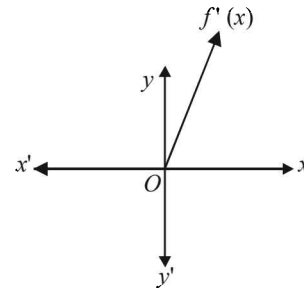
$\therefore f(x)$  is not continuous on  $(0, \pi)$ .

10. (b,c,d)  $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

$\therefore f'(x) = \begin{cases} 0, & x < 0 \\ 2x, & x \geq 0 \end{cases}$

which exists  $\forall x$  except possibly at  $x = 0$ .

At  $x = 0$ ,  $Lf' = 0 = Rf' \Rightarrow f$  is differentiable.



From graph of  $f'(x)$ , it is clear that  $f'(x)$  is continuous but not differentiable at  $x = 0$ .

11. (a, b) We have  $g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

If  $x \neq 0$ ,  $g'(x) = x^2 \cos(1/x) \left(-\frac{1}{x^2}\right) + 2x \sin \frac{1}{x}$

=  $-\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$

which exists for  $\forall x \neq 0$ .

If  $x = 0$ ,

$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0}$

=  $\lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$

$\Rightarrow g'(x) = \begin{cases} -\cos\left(\frac{1}{x}\right) + 2x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

At  $x = 0$ ,  $\cos\left(\frac{1}{x}\right)$  is not continuous, therefore  $g'(x)$  is not continuous at  $x = 0$ .

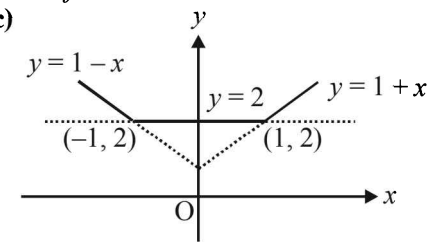
At  $x = 0$

$Lf' = \lim_{x \rightarrow 0} \frac{0 - (-x) \sin\left(-\frac{1}{x}\right)}{x} = -\sin\left(\frac{1}{x}\right)$

which does not exist.

$\therefore f$  is not differentiable at  $x = 0$ .

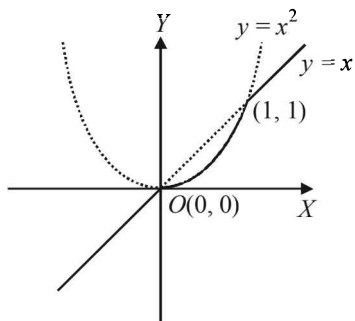
12. (a, c)



From graph it is clear that  $f(x)$  is continuous every where and also differentiable everywhere except at  $x = 1$  and  $-1$ .

13. (a, c, d) From the figure it is clear that

$$h(x) = \begin{cases} x, & \text{if } x \leq 0 \\ x^2, & \text{if } 0 < x < 1 \\ x, & \text{if } x \geq 1 \end{cases}$$



From the graph it is clear that  $h$  is continuous for all  $x \in \mathbb{R}$ ,  $h'(x) = 1$  for all  $x > 1$  and  $h$  is not differentiable at  $x = 0$  and  $1$ .

14. (d) L.H.L. =  $\lim_{x \rightarrow 1^-} \frac{\sqrt{1 - \cos[2(x-1)]}}{x-1}$

$$= \lim_{x \rightarrow 1^-} \frac{\sqrt{2 \sin^2(x-1)}}{x-1} = \sqrt{2} \cdot \lim_{x \rightarrow 1^-} \frac{\sqrt{\sin^2(x-1)}}{x-1}$$

$$= \sqrt{2} \lim_{x \rightarrow 1^-} \frac{|\sin(x-1)|}{x-1} = \sqrt{2} \lim_{h \rightarrow 0} \frac{|\sin(-h)|}{-h}$$

$$= \sqrt{2} \lim_{h \rightarrow 0} \frac{\sin h}{-h} = -\sqrt{2}$$

Again,

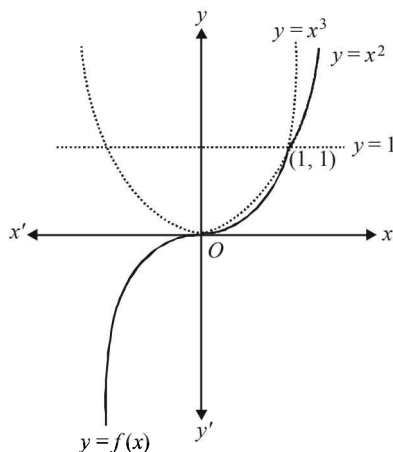
R.H.L. =  $\lim_{x \rightarrow 1^+} \frac{\sqrt{1 - \cos(x-1)}}{x-1} = \lim_{x \rightarrow 1^+} \sqrt{2} \frac{|\sin(x-1)|}{x-1}$

Put  $x = 1 + h, h > 0$  for  $x \rightarrow 1^+, h \rightarrow 0$ .

$$= \lim_{h \rightarrow 0} \sqrt{2} \frac{|\sin h|}{h} = \lim_{h \rightarrow 0} \sqrt{2} \frac{\sin h}{h} = \sqrt{2}$$

L.H.L.  $\neq$  R.H.L. Therefore  $\lim_{x \rightarrow 1} f(x)$  does not exist.

15. (a, d) From graph,  $f(x)$  is continuous everywhere but not differentiable at  $x = 1$ .



16. (a,c) Given that  $L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, a > 0$

and  $L$  is finite.

Now  $L = \lim_{x \rightarrow 0} \frac{\frac{x}{\sqrt{a^2 - x^2}} - \frac{x}{2}}{4x^3}$  (Using L'Hospital's rule)

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{a^2 - x^2}} - \frac{1}{2}}{4x^2}$$

$\therefore L$  is finite, limiting value of numerator should be

zero which is so when  $\frac{1}{\sqrt{a^2}} - \frac{1}{2} = 0$

i.e.  $a = 2$  ( $\because a > 0$ )

Applying L'Hospital's rule again, we get

$$L = \lim_{x \rightarrow 0} \frac{\frac{x}{(a^2 - x^2)^{3/2}}}{8x} = \lim_{x \rightarrow 0} \frac{1}{8(a^2 - x^2)^{3/2}}$$

$$= \frac{1}{8 \times a^3} = \frac{1}{8 \times 8} \quad (\text{using } a = 2)$$

$$= \frac{1}{64}$$

17. (b, c)  $\because f(x+y) = f(x) + f(y) \forall x, y \in \mathbb{R}$

$\therefore$  Putting  $x = y = 0$ , we get

$$f(0) = 0$$

Also  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{f(h)}{h} = f'(0) = k \text{ (say)}$$

$$\Rightarrow f(x) = kx + c$$

But  $f(0) = 0 \Rightarrow c = 0$

$$\therefore f(x) = kx$$

Which is continuous and differentiable  $\forall x \in \mathbb{R}$ .

$\therefore$  b and c are the correct options.

18. (a, b, c, d)

At  $x = -\frac{\pi}{2}$

LHL =  $\lim_{x \rightarrow \frac{\pi^-}{2}} -x - \frac{\pi}{2} = 0$

RHL =  $\lim_{x \rightarrow \frac{\pi^+}{2}} -\cos x = 0$  and  $f\left(\frac{-\pi}{2}\right) = 0$

$$\therefore \text{LHL} = \text{RHL} = f\left(\frac{-\pi}{2}\right)$$

$$\Rightarrow f(x) \text{ is continuous at } x = -\frac{\pi}{2}$$

Also at  $x = 0$

$$L f'(0) = \sin 0 = 0; R f'(0) = 1 - 0 = 1$$

$$\therefore L f'(0) \neq R f'(0)$$

$\Rightarrow f$  is not differentiable at  $x = 0$

At  $x = 1$

$$L f'(1) = R f'(1) \Rightarrow f \text{ is differentiable at } x = 1.$$

At  $x = \frac{-3}{2}, f(x) = -\cos x$  which is differentiable.

$\therefore$  All four options are correct.

## Limits, Continuity and Differentiability

19. (b,d)

$$\text{We have } f(x) = \begin{cases} a_n + \sin \pi x, & x \in [2n, 2n+1] \\ b_n + \cos \pi x, & x \in (2n-1, 2n) \end{cases}$$

As  $f$  is continuous for all  $n$  $\therefore$  At  $x = 2n$ , LHL = RHL =  $f(2n)$ 

$$\Rightarrow b_n + \cos 2\pi n = a_n + \sin 2\pi n = a_n + \sin 2\pi n$$

$$\Rightarrow b_n + 1 = a_n \Rightarrow a_n - b_n = 1$$

 $\therefore b$  is correct.Also at  $x = 2n + 1$ , LHL = RHL =  $f(2n + 1)$ 

$$\Rightarrow \lim_{h \rightarrow 0} a_n + \sin \pi(2n + 1 - h)$$

$$= \lim_{h \rightarrow 0} b_{n+1} + \cos \pi(2n + 1 - h) = a_n + \sin(2n + 1)\pi$$

$$\Rightarrow a_n = b_{n+1} - 1 = a_n \Rightarrow a_n - b_{n+1} = -1$$

 $\therefore c$  is incorrect

$$\Rightarrow a_{n-1} - b_n = -1$$

 $\therefore d$  is correct.

20. (b,d)

$$\lim_{n \rightarrow \infty} \frac{1^a + 2^a + \dots + n^a}{(n+1)^{a-1} [(na+1) + (na+2) + \dots + (na+n)]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^a \left( \left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \dots + \left(\frac{n}{n}\right)^a \right)}{(n+1)^{a-1} \left( n^2 a + \frac{n(n+1)}{2} \right)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^{a-1} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^a}{(n+1)^{a-1} \left( a + \frac{1}{2} \left(1 + \frac{1}{n}\right) \right)} = \frac{1}{60}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left( \frac{1}{1 + \frac{1}{n}} \right)^{a-1} \frac{\frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^a}{a + \frac{1}{2} \left(1 + \frac{1}{n}\right)} = \frac{1}{60}$$

$$\Rightarrow \frac{\int_0^1 x^a dx}{a + \frac{1}{2}} = \frac{1}{60} \Rightarrow \frac{\left[ \frac{x^{a+1}}{a+1} \right]_0^1}{\left( a + \frac{1}{2} \right)} = \frac{1}{60}$$

$$\Rightarrow \frac{1}{(a+1) \left( a + \frac{1}{2} \right)} = \frac{1}{60}$$

$$\Rightarrow 2a^2 + 3a - 119 = 0 \Rightarrow a = 7 \text{ or } -\frac{17}{2}$$

21. (a,c)  $g(x)$  may be discontinuous at  $x = a$  or  $x = b$ .Let us check the continuity of  $g(x)$  at  $x = a$  and  $x = b$ 

$$\lim_{x \rightarrow a^-} g(x) = 0$$

$$\lim_{x \rightarrow a^+} g(x) = \lim_{x \rightarrow a^+} \int_a^x f(t) dt = \int_a^a f(t) dt = 0$$

$$g(a) = \int_a^a f(t) dt = 0$$

 $\therefore g(x)$  is continuous at  $x = a$ 

$$\text{Also } \lim_{x \rightarrow b^-} g(x) = \lim_{x \rightarrow b^-} \int_a^x f(t) dt = \int_a^b f(t) dt$$

$$\lim_{x \rightarrow b^+} g(x) = \int_a^b f(t) dt \Rightarrow g(b) = \int_a^b f(t) dt$$

 $\therefore g(x)$  is continuous at  $x = b$ Hence  $g(x)$  is continuous  $\forall x \in R$ 

$$\text{Now } g'(x) = \begin{cases} 0, & x < a \\ f(x), & a \leq x \leq b \\ 0, & x > b \end{cases}$$

$$g'(a^-) = 0 \text{ and } g'(a^+) = f(a)$$

$$g'(b^-) = f(b) \text{ and } g'(b^+) = 0$$

As  $f(a), f(b) \in [1, \infty) \therefore f(a), f(b) \neq 0$ Hence  $g'(a^-) \neq g'(a^+)$  and  $g'(b^-) \neq g'(b^+)$  $\therefore g$  is not differentiable at  $a$  and  $b$ .22. (a, d) Let  $f$  and  $g$  be maximum at  $c_1$  and  $c_2$  respectively,

$$c_1, c_2 \in (0, 1)$$

$$\text{Then, } f(c_1) = g(c_2)$$

$$\text{Let } h(x) = f(x) - g(x)$$

$$\text{Then, } h(c_1) = f(c_1) - g(c_1) > 0$$

$$\text{and } h(c_2) = f(c_2) - g(c_2) < 0$$

 $\therefore h(x) = 0$  has at least one root in  $(c_1, c_2)$ 

$$C \in (c_1, c_2) \text{ i.e. for } f(c) = g(c)$$

which shows that (a) and (d) are correct.

$$23. \text{ (a, d)} f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} -g(x), & x < 0 \\ 0, & x = 0 \\ g(x), & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -g'(x), & x < 0 \\ 0, & x = 0 \\ g'(x), & x > 0 \end{cases}$$

$$\therefore Lf'(0) = -g'(0) = 0$$

$$Rf'(0) = g'(0) = 0$$

 $\therefore f$  is differentiable at  $x = 0$ 

$$h(x) = e^{|x|} = \begin{cases} e^{-x}, & x < 0 \\ e^x, & x \geq 0 \end{cases}$$

$$h'(x) = \begin{cases} -e^{-x}, & x < 0 \\ e^x, & x \geq 0 \end{cases} \Rightarrow Lh'(0) = -1, Rh'(0) = 1$$

 $\therefore h$  is not differentiable at  $x = 0$ 

$$f \circ h(x) = f(h(x)) = g(e^{|x|}) \text{ as } e^{|x|} > 0$$

$$= \begin{cases} g(e^{-x}) & \text{if } x < 0 \\ g(1) & \text{if } x = 0 \\ g(e^x) & \text{if } x > 0 \end{cases}$$

$$f'[h(x)] = \begin{cases} -g'(e^{-x})e^{-x} & , x < 0 \\ 0 & , x = 0 \\ g'(e^x)e^x & , x > 0 \end{cases}$$

$$Lf'(h(0)) = -g'(1), Rf'(h(0)) = g'(1)$$

$$\therefore g'(1) \neq 0, \therefore Lf'(h(0)) \neq Rf'(h(0))$$

$\therefore f \circ h$  is not differentiable at  $x = 0$ .

$$hof(x) = \begin{cases} e^{|f(x)|} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

$$Lh'(f(0)) = \lim_{k \rightarrow 0} \frac{h(f(0)) - h(f(0-k))}{k}$$

$$= \lim_{k \rightarrow 0} \frac{1 - e^{|g(-k)|}}{k} = \lim_{k \rightarrow 0} \frac{1 - e^{|g(-k)|}}{|g(-k)|} \times \frac{|g(-k)|}{k}$$

$$= 1 \times 0 = 0 \left( \because g'(0) = 0 \Rightarrow \lim_{k \rightarrow 0} \frac{g(-k)}{k} = \lim_{k \rightarrow 0} \frac{g(k)}{k} = 0 \right)$$

$$Rh'(f(0)) = \lim_{k \rightarrow 0} \frac{h(f(0+k)) - h(f(0))}{k}$$

$$= \lim_{k \rightarrow 0} \frac{e^{|g(k)|} - 1}{k} = \lim_{k \rightarrow 0} \frac{e^{|g(k)|} - 1}{|g(k)|} \times \frac{|g(k)|}{k} = 0$$

$\therefore h \circ f$  is differentiable at  $x = 0$ .

24. (a, b)  $f(x) = a \cos^3(|x^3 - x|) + b|x| \sin(|x^3 + x|)$

(a) If  $a = 0, b = 1$

$$\Rightarrow f(x) = |x| \sin |x^3 + x|$$

$$= x \sin(x^3 + x), x \in \mathbb{R}$$

$\therefore f$  is differentiable every where.

(b), (c) If  $a = 1, b = 0 \Rightarrow f(x) = \cos^3(|x^3 - x|)$   
 $= \cos^3(x^3 - x)$

which is differentiable every where.

(d) when  $a = 1, b = 1, f(x) = \cos(x^3 - x) + x \sin(x^3 + x)$   
 which is differentiable at  $x = 1$

$\therefore$  Only a and b are the correct options.

25. (b, c)  $f(x) = [x^2 - 3]$  is discontinuous at all integral points in

$$\left[-\frac{1}{2}, 2\right]$$

Which happens when  $x = 1, \sqrt{2}, \sqrt{3}, 2$

$\therefore f$  is discontinuous exactly at four points in  $\left[-\frac{1}{2}, 2\right]$

$$\text{Also } g(x) = (|x| + |4x - 7|)f(x)$$

Here  $f$  is not differentiable at  $x = 1, \sqrt{2}, \sqrt{3} \in \left(-\frac{1}{2}, 2\right)$

and  $|x| + |4x - 7|$  is not differentiable at 0 and  $\frac{7}{4}$

$$\text{But } f(x) = 0, \forall x \in \left[\sqrt{3}, 2\right]$$

$\therefore g(x)$  becomes differentiable at  $x = \frac{7}{4}$

Hence  $g(x)$  is non-differentiable at four points i.e.,  $0, 1, \sqrt{2}, \sqrt{3}$

E. SUBJECTIVE PROBLEMS

1. 
$$\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$$

$$= \lim_{x \rightarrow a} \frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{a+2x} + \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{3a+x} - 2\sqrt{x})(\sqrt{3a+x} + 2\sqrt{x})(\sqrt{a+2x} + \sqrt{3x})}$$

$$= \lim_{x \rightarrow a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$= \lim_{x \rightarrow a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{3(a-x)(\sqrt{a+2x} + \sqrt{3x})}$$

$$= \lim_{x \rightarrow a} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{3(\sqrt{a+2x} + \sqrt{3x})} = \frac{\sqrt{3a+a} + 2\sqrt{a}}{3(\sqrt{a+2a} + \sqrt{3a})}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

NOTE : The given limit is of the form  $\frac{0}{0}$ . Hence limit of the function can also be find out by using L' Hospital's Rule.

2. 
$$f(x) = \int \frac{2 \sin x - \sin 2x}{x^3} dx, x \neq 0$$

$$\therefore f'(x) = \frac{2 \sin x - \sin 2x}{x^3}, x \neq 0$$

$$\therefore \lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin x(1 - \cos x)(1 + \cos x)}{x^3(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} 2 \cdot \frac{\sin^3 x}{x^3} \cdot \frac{1}{1 + \cos x}$$

$$= 2 \times (1)^3 \times \frac{1}{2} = 1$$

3. 
$$\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^2[\sin(a+h) - \sin a] + 2ah \sin(a+h) + h^2 \sin(a+h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^2 \left[ 2 \cos \left( a + \frac{h}{2} \right) \sin \frac{h}{2} \right] + 2a \sin(a+h)}{2 \times \frac{h}{2}} + h \sin(a+h)$$

$$= a^2 \cos a + 2a \sin a$$

4. As  $f(x)$  is continuous at  $x = 0$ , we have  
 $LHL = RHL = f(0)$   
 $\Rightarrow \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(0+h) = f(0)$   
 $\Rightarrow f(0) + \lim_{h \rightarrow 0} f(-h) = f(0) + \lim_{h \rightarrow 0} f(h) = f(0)$   
 [Using the given property  $f(x+y) = f(x) + f(y)$ ]  
 $\Rightarrow \lim_{h \rightarrow 0} f(-h) = \lim_{h \rightarrow 0} f(h) = 0 \dots (1)$

## Limits, Continuity and Differentiability

Now let  $x = a$  be any arbitrary point then at  $x = a$ ,

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} [f(a) + f(-h)] \\ &= f(a) + \lim_{h \rightarrow 0} f(-h) = f(a) \quad [\text{Using, } f(x+y) = f(x) + f(y)] \\ &= f(a) + \lim_{h \rightarrow 0} f(-h) = f(a) \quad [\text{using eq}^n(1)] \end{aligned}$$

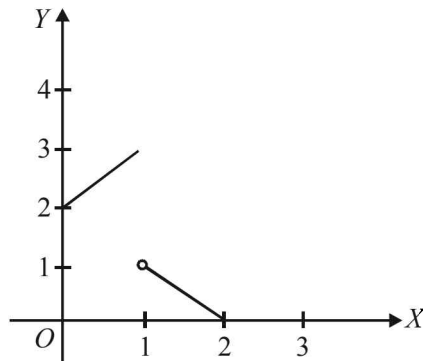
$$\text{Similarly, R.H.L.} = \lim_{h \rightarrow 0} f(a+h) = f(a)$$

Thus, we get

$$\begin{aligned} \lim_{h \rightarrow 0} f(a-h) &= \lim_{h \rightarrow 0} f(a+h) = f(a) \\ \Rightarrow f &\text{ is continuous at } x = a. \text{ But } a \text{ is any arbitrary point} \\ \therefore f &\text{ is continuous } \forall x \in R. \end{aligned}$$

$$\begin{aligned} 5. \quad \lim_{x \rightarrow 0} \frac{2^x - 1}{\sqrt{1+x} - 1} &= \lim_{x \rightarrow 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} \\ &= \lim_{x \rightarrow 0} \frac{(2^x - 1)(\sqrt{1+x} + 1)}{1 + x - 1} \\ &= \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \cdot \lim_{x \rightarrow 0} (\sqrt{1+x} + 1) \\ &= \ln 2 \cdot (1 + 1) = 2 \ln 2. \end{aligned}$$

6. Graph of  $f(x)$  is



Clearly from graph  $f(x)$  is discontinuous at  $x = 1$  and  $2$ .

$$7. \quad \text{We have } f(x) = \frac{x^2}{2}, 0 \leq x < 1 = 2x^2 - 3x + \frac{3}{2}, 1 \leq x \leq 2$$

Here  $f(x)$  is continuous everywhere except possibly at  $x = 1$

$$\Rightarrow \text{At } x = 1, Lf' = \frac{2}{2} \times 1 = 1; Rf' = 4 \times 1 - 3 = 1$$

$\Rightarrow f$  is differentiable and hence continuous at  $x = 1$

$\therefore f(x)$  is continuous on  $[0, 2]$

$$\begin{aligned} f'(x) &= x, 0 \leq x < 1 \\ &= 4x - 3, 1 \leq x \leq 2 \end{aligned}$$

At  $x = 1$ ,

$$\lim_{x \rightarrow 1^-} f'(x) = \lim_{h \rightarrow 0} f'(1-h) = \lim_{h \rightarrow 0} (1-h) = 1$$

$$\lim_{x \rightarrow 1^+} f'(x) = \lim_{h \rightarrow 0} f'(1+h) = \lim_{h \rightarrow 0} 4(1+h) - 3 = 1$$

$$\begin{aligned} f'(1) &= 4 - 3 = 1 \\ \therefore f' &\text{ is continuous at } x = 1 \end{aligned}$$

$\therefore f'$  is continuous on  $[0, 2]$

$$f''(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 4, & 1 \leq x \leq 2 \end{cases}$$

Clearly  $f''(x)$  is discontinuous at  $x = 1$ ,

$\therefore f''(x)$  is discontinuous on  $[0, 2]$ .

Given  $f(x) = x^3 - x^2 + x + 1$

$$\begin{aligned} 8. \quad \therefore f'(x) &= 3x^2 - 2x + 1 = 3 \left( x^2 - \frac{2}{3}x + \frac{1}{3} \right) \\ &= 3 \left[ \left( x - \frac{1}{3} \right)^2 - \frac{1}{9} + \frac{1}{3} \right] \\ &= 3 \left[ \left( x - \frac{1}{3} \right)^2 + \frac{2}{9} \right] > 0 \forall x \in R. \end{aligned}$$

Hence  $f(x)$  is an increasing function of  $x$  for all real values of  $x$ .

Now  $\max [f(t) : 0 \leq t \leq x]$  means the greatest value of

$f(t)$  in  $0 \leq t \leq x$  which is obtained at  $t = x$ , since  $f(t)$  is increasing for all  $t$ .

$$\therefore \max. [f(t) : 0 \leq t \leq x] = x^3 - x^2 + x + 1$$

Hence the function  $g$  is defined as follows :

$$\begin{aligned} g(x) &= x^3 - x^2 + x + 1 \quad \text{when } 0 \leq x \leq 1 \\ &= 3 - x \quad \text{when } 1 < x \leq 2 \end{aligned}$$

Now it is sufficient to discuss the continuity and differentiability of  $g(x)$  at  $x = 1$ . Since for all other values of  $x$ ,  $g(x)$  is clearly continuous and differentiable, being a polynomial function of  $x$ .

We have,  $g(1) = 2$

$$g(1-0) = \lim_{h \rightarrow 0} [(1-h)^3 - (1-h)^2 + (1-h) + 1] = 2$$

$$g(1+0) = \lim_{h \rightarrow 0} [3 - (1+h)] = 2$$

Hence  $g(x)$  is continuous at  $x = 1$

Now,

$$Lg'(1) = \lim_{h \rightarrow 0} \frac{[(1-h)^3 - (1-h)^2 + (1-h) + 1] - 2}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{1 - 3h + 3h^2 - h^3 - 1 + 2h - h^2 + 1 - h + 1 - 2}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h + 2h^2 - h^3}{-h} = \lim_{h \rightarrow 0} [2 - 2h + h^2] = 2$$

$$Rg'(1) = \lim_{h \rightarrow 0} \frac{[3 - (1+h) - 2]}{h} = \lim_{h \rightarrow 0} \frac{-h}{h} = -1$$

Since  $Lg'(1) \neq Rg'(1)$ , the function  $g(x)$  is not differentiable at  $x = 1$

Hence  $g(x)$  is continuous on  $(0, 2)$ . It is also differentiable on  $(0, 2)$  except at  $x = 1$ .

We have  $f(x) = -1, -2 \leq x \leq 0$

$$= x - 1, 0 < x \leq 2$$

and  $g(x) = f(|x| + |f(x)|)$

Hence  $g(x)$  involves  $|x|$  and  $|x-1|$  or  $|-1| = 1$

Therefore we should divide the given interval  $(-2, 2)$  into the following intervals.



$I_1$	$I_2$	$I_3$
$[-2, 2] = [-2, 0)$	$[0, 1)$	$[1, 2]$
$x = -ve$	$+ve$	$+ve$
$ x  = -x$	$x$	$x$
$f(x) = -1$	$x - 1$	$x - 1$
$f( x ) = -1$	$= x - 1$	$= x - 1$
$ f(x)  =  -1 $	$ x - 1 $	$ x - 1 $
$= 1$	$= -(x - 1)$	$= x - 1$

∴ Using above we get

$$g(x) = f(|x|) + |f(x)|$$

$$= -1 + 1 = 0 \text{ in } I_1$$

$$= x - 1 - (x - 1) = 0 \text{ in } I_2$$

$$= x - 1 + x - 1 = 2(x - 1) \text{ in } I_3$$

Hence  $g(x)$  is defined as follows :

$$g(x) = \begin{cases} 0, & -2 \leq x < 1 \\ 2(x - 1), & 1 \leq x \leq 2 \end{cases}$$

$Lg'(1) = 0; Rg'(1) = 2$  (not equal)

Hence  $g(x)$  is not differentiable at  $x = 1$ .

10.

Let  $h(x) = f(x) + g(x)$  be continuous.

Then,  $g(x) = h(x) - f(x)$

Now,  $h(x)$  and  $f(x)$  both are continuous functions.

∴  $h(x) - f(x)$  must also be continuous. But it is a contradiction as given that  $g(x)$  is discontinuous. Therefore our assumption that  $f(x) + g(x)$  a continuous function is wrong and hence  $f(x) + g(x)$  is discontinuous.

11.

Given that  $f(x)$  is a function satisfying

$$f(-x) = f(x), \forall x \in R \quad \dots(1)$$

Also  $f'(0)$  exists

$$\Rightarrow f'(0) = Rf'(0) = Lf'(0)$$

Now,  $Rf'(0) = f'(0)$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0)$$

Again  $Lf'(0) = f'(0) \quad \dots(2)$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0) \quad \dots(3)$$

[Using eq. (1)]

From equations (2) and (3), we get

$$\Rightarrow f'(0) = 0$$

12.

Given that,

$$f(x) = \begin{cases} x + a\sqrt{2} \sin x & , 0 \leq x < \frac{\pi}{4} \\ 2x \cot x + b & , \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ a \cos 2x - b \sin x & , \frac{\pi}{2} < x \leq \pi \end{cases}$$

is continuous for  $0 \leq x \leq \pi$ .

$$\therefore f(x) \text{ must be continuous at } x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow \left(\frac{\pi}{4}\right)^-} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \lim_{h \rightarrow 0} f\left(\frac{\pi}{4} - h\right) = \frac{2\pi}{4} \cot \frac{\pi}{4} + b$$

$$\Rightarrow \lim_{h \rightarrow 0} \left(\frac{\pi}{4} - h\right) + a\sqrt{2} \sin\left(\frac{\pi}{4} - h\right) = \frac{\pi}{2} + b$$

$$\Rightarrow \frac{\pi}{4} + a = \frac{\pi}{2} + b$$

$$\Rightarrow a - b = \frac{\pi}{4} \quad \dots(1)$$

Also,  $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^+} f(x) = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{h \rightarrow 0} f\left(\frac{\pi}{2} + h\right) = 2 \cdot \frac{\pi}{2} \cot \frac{\pi}{2} + b$$

$$\Rightarrow \lim_{h \rightarrow 0} a \cos 2\left(\frac{\pi}{2} + h\right) - b \sin\left(\frac{\pi}{2} + h\right) = b$$

$$\Rightarrow a \cos \pi - b \sin \frac{\pi}{2} = b \Rightarrow -a - b = b$$

$$\Rightarrow a + 2b = 0 \quad \dots(2)$$

Solving (1) and (2), we get  $a = \frac{\pi}{6}$  and  $b = -\frac{\pi}{12}$ .

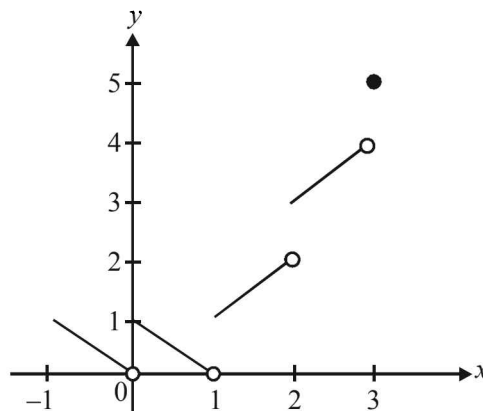
We have,  $[x] + |1 - x|, -1 \leq x \leq 3$

**NOTE THIS STEP**

$$\text{or } y = \begin{cases} -1 + 1 - x & , -1 \leq x < 0 \\ 0 + 1 - x & , 0 \leq x < 1 \\ 1 - 1 + x & , 1 \leq x < 2 \\ 2 - 1 + x & , 2 \leq x < 3 \\ 3 - 1 + x & , x = 3 \end{cases}$$

$$\text{or } y = \begin{cases} -x & , -1 \leq x < 0 \\ 1 - x & , 0 \leq x < 1 \\ x & , 1 \leq x < 2 \\ 1 + x & , 2 \leq x < 3 \\ 2 + x & , x = 3 \end{cases}$$

From graph we can say that given functions is not differentiable at  $x = 0, 1, 2, 3$ .



## Limits, Continuity and Differentiability

14. We are given that,

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & , x < 0 \\ a & , x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & , x > 0 \end{cases}$$

Here L.H.L at  $(x=0)$

$$= \lim_{h \rightarrow 0} \frac{1 - \cos 4(0-h)}{(0-h)^2} = \lim_{h \rightarrow 0} \frac{1 - \cos 4h}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{2 \sin^2 2h}{4h^2} \cdot 4 = 8$$

R.H.L at  $(x=0)$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{0+h}}{\sqrt{16+\sqrt{0+h}}-4} = \lim_{h \rightarrow 0} \frac{\sqrt{h}(\sqrt{16+h}+4)}{16+\sqrt{h}-16}$$

$$= \lim_{h \rightarrow 0} \sqrt{16+\sqrt{h}}+4 = \sqrt{16}+4 = 8$$

For continuity of function  $f(x)$ , we must have

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

$$\Rightarrow f(0) = 8 \Rightarrow a = 8$$

15. We are given

$$f(x+y) = f(x)f(y), \forall x, y \in R$$

$$f(x) \neq 0, \text{ for any } x$$

$$f \text{ is differentiable at } x=0, f'(0) = 2$$

To prove that  $f'(x) = 2f(x)$ ,  $\forall x \in R$  and to find  $f(x)$ .

We have for  $x = y = 0$

$$f(0+0) = f(0)f(0)$$

$$\Rightarrow f(0) = [f(0)]^2 \Rightarrow f(0) = 1$$

Again  $f'(0) \neq 2$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 2 \Rightarrow \lim_{h \rightarrow 0} \frac{f(0)f(h) - f(0)}{h} = 2$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(0)[f(h)-1]}{h} = 2$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(h)-1}{h} = 2 \quad \dots(1) \quad [\text{Using } f(0) = 1]$$

$$\text{Now, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} = \lim_{h \rightarrow 0} f(x) \left( \frac{f(h)-1}{h} \right)$$

$$= f(x) \lim_{h \rightarrow 0} \left[ \frac{f(h)-1}{h} \right]$$

$$= f(x) \cdot 2 \quad [\text{Using eq. (1)}]$$

$$= 2f(x)$$

$$\text{Also, } \frac{f'(x)}{f(x)} = 2$$

Integrating on both sides with respect to  $x$ , we get

$$\log |f(x)| = 2x + C$$

$$\text{At } x=0, \log f(0) = C \Rightarrow C = \log 1 = 0$$

$$\therefore \log |f(x)| = 2x \Rightarrow f(x) = e^{2x}$$

16.

$$\lim_{x \rightarrow 0} \left\{ \tan\left(\frac{\pi}{4}\right) + x \right\}^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \log \left\{ \tan\left(\frac{\pi}{4}\right) + x \right\}^{\frac{1}{x}}}$$

$$[\text{Using } \lim_{x \rightarrow a} f(x) = e^{\lim_{x \rightarrow a} \log f(x)}]$$

$$= e^{\lim_{x \rightarrow 0} \frac{\log \tan\left(\frac{\pi}{4}\right) + x}{x}} \quad \left[ \frac{0}{0} \text{ form} \right]$$

$$= e^{\lim_{x \rightarrow 0} \left[ \frac{\sec^2\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4}\right) + x} \right]}$$

[Using L' Hospital's rule]

$$= e^{\frac{2}{1}} = e^2$$

17.

$$\text{Given that, } f(x) = \begin{cases} (1 + |\sin x|)^{\frac{a}{|\sin x|}} & -\frac{\pi}{6} < x < 0 \\ b & x = 0 \\ e^{\tan 3x} & 0 < x < \frac{\pi}{6} \end{cases}$$

is continuous at  $x = 0$

$$\therefore \lim_{h \rightarrow 0} f(0-h) = f(0) = \lim_{h \rightarrow 0} f(0+h)$$

We have,

$$\lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} [1 + |\sin(-h)|]^{\frac{a}{|\sin(-h)|}}$$

$$= \lim_{h \rightarrow 0} [1 + \sin h]^{\frac{a}{\sin h}}$$

$$\lim_{h \rightarrow 0} \frac{a}{\sin h} \log(1 + \sin h) = e^a$$

$$\text{and } f(0) = b$$

$$\therefore e^a = b \quad \dots(1)$$

$$\text{Also } \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} e^{\frac{\tan 2h}{\tan 3h}}$$

$$= e^{\lim_{h \rightarrow 0} \frac{\tan 2h}{2h} \times \frac{3h}{\tan 3h} \times \frac{2}{3}} = e^{\frac{2}{3}}$$

$$\therefore e^{\frac{2}{3}} = b \quad \dots(2)$$

From (1) and (2)

$$e^a = b = e^{\frac{2}{3}} \Rightarrow a = \frac{2}{3} \text{ and } b = e^{\frac{2}{3}}$$

18.

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \dots(1)$$

Putting  $y = 0$  and  $f(0) = 1$  in (1), we get

$$f\left(\frac{x}{2}\right) = \frac{1}{2}[f(x)+1]$$

$$\therefore f(x) = 2f\left(\frac{x}{2}\right) - 1 \quad \dots(2)$$

$$\text{Now, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{f(2x) + f(2h)}{2} - f(x) \right], \text{ by (1)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{(2f(x) - 1) + (2f(h) - 1)}{2} - f(x) \right], \\
 &\hspace{15em} \text{by (2)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [f(h) - 1] \\
 &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0) = -1
 \end{aligned}$$

Hence  $f'(x) = -1$ , integrating, we get

$f(x) = -x + c$ . Putting  $x = 0$ , we get

$f(0) = c = 1$  by (1)  $\therefore f(x) = 1 - x$

$f(2) = 1 - 2 = -1$

19. By the given definition it is clear that the function  $f$  is continuous and differentiable at all points except possible at  $x = 1$  and  $x = 2$ .

Continuity at  $x = 1$

$$\text{L.H.L.} = \lim_{h \rightarrow 0} [1 - (1 - h)] = \lim_{h \rightarrow 0} h = 0$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} [1 - (1 + h)][2 - (1 + h)]$$

$$= \lim_{h \rightarrow 0} \{-h(1 - h)\} = 0$$

Also,  $f(1) = 0$

$\therefore$  L.H.L. = R.H.L. =  $f(1) = 0$

Therefore,  $f$  is continuous at  $x = 1$

Now, differentiability at  $x = 1$

$$\begin{aligned}
 Lf'(1) &= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}, h > 0 \\
 &= \lim_{h \rightarrow 0} \frac{(1 - (1-h)) - 0}{-h} = \lim_{h \rightarrow 0} \left( \frac{h}{-h} \right) = -1
 \end{aligned}$$

$$\begin{aligned}
 \text{and } Rf'(1) &= \lim_{h \rightarrow 0} \frac{f((1+h) - f(1))}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\{1 - (1-h)\} \{2 - (1-h)\} - 0}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-h(1-h)}{h} = \lim_{h \rightarrow 0} (h-1) = -1
 \end{aligned}$$

Since  $Lf'(1) = Rf'(1)$

Hence,  $f$  is differentiable at  $x = 1$

Continuity at  $x = 2$

$$\text{L.H.L.} = \lim_{h \rightarrow 0} [1 - (2 - h)][2 - (2 - h)]$$

$$\lim_{h \rightarrow 0} \{-1 + h\} \{h\} = 0$$

$$\text{and R.H.L.} = \lim_{h \rightarrow 0} [3 - (2 + h)] = \lim_{h \rightarrow 0} (1 - h) = 1$$

Since L.H.L.  $\neq$  R.H.L., therefore  $f$  is not continuous at  $x = 2$ . As such  $f$  cannot be differentiable at  $x = 2$ . Hence  $f$  is continuous and differentiable at all points except at  $x = 2$ .

20. Given that,  $F(x) = \int_0^x f(t) dt$

**NOTE THIS STEP**

$$\therefore F'(x) = f(x) \cdot 1 - f(0) \cdot 0$$

[Using Leibnitz theorem]

$$\Rightarrow F'(x) = f(x) \dots (1), \forall x \geq 0$$

$$\text{Also } F(0) = \int_0^0 f(t) dt = 0$$

But given that  $f(x) \leq cF(x), \forall x \geq 0$

$$\therefore \text{ We get } f(0) \leq cF(0) = 0$$

$$\therefore f(0) \leq 0 \dots (2)$$

But  $ATQ f(x)$  is non-negative continuous function on  $[0, \infty)$

$$\therefore f(x) \geq 0$$

$$\therefore f(0) \geq 0 \dots (3)$$

$$\therefore \text{ From (2) and (3) } f(0) = 0$$

Again  $f(x) \leq cF(x) \forall x \geq 0$ , we get

$$f(x) - cF(x) \leq 0$$

$$\Rightarrow F'(x) - cF(x) \leq 0, \forall x \geq 0 \text{ [Using equation (1)]}$$

$$e^{cx} F'(x) - ce^{-cx} F(x) \leq 0$$

[Multiplying both sides by  $e^{-cx}$  (I.F.) and keeping in mind that  $e^{-cx} > 0, \forall x$ ]

$$\Rightarrow \frac{d}{dx} [e^{-cx} F(x)] \leq 0$$

$$\Rightarrow g(x) = e^{-cx} F(x) \text{ is a decreasing function on } [0, \infty).$$

That is  $g(x) \leq g(0)$  for all  $x \geq 0$

But  $g(0) = F(0) = 0$

$$\therefore g(x) \leq 0, \forall x \geq 0$$

$$\Rightarrow e^{-cx} F(x) \leq 0, \forall x \geq 0$$

$$\Rightarrow F(x) \leq 0, \forall x \geq 0$$

$$\therefore f(x) \leq cF(x) \leq 0, \forall x \geq 0$$

$$[\because c > 0 \text{ and using } f(x) \leq cF(x)]$$

$$\Rightarrow f(x) \leq 0, \forall x \geq 0$$

But given  $f(x) \geq 0$

$$\Rightarrow f(x) = 0, \forall x \geq 0.$$

21. (I)  $g$  is continuous at  $\alpha$  and

$$f(x) - f(\alpha) = g(x)(x - \alpha), \forall x \in R$$

$\Rightarrow$  Since  $g$  is continuous at  $x = \alpha$

$$\text{and } g(x) = \frac{f(x) - f(\alpha)}{x - \alpha}$$

We should have,  $\lim_{x \rightarrow \alpha} g(x) = g(\alpha)$

$$\Rightarrow \lim_{x \rightarrow \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = g(\alpha) \Rightarrow f'(\alpha) = g(\alpha)$$

$\Rightarrow f'(\alpha)$  exists and is equal to  $g(\alpha)$ .

- (II)  $\therefore f(x)$  is differentiable at  $x = \alpha$

$$\therefore \lim_{x \rightarrow \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha)$$

exists and is finite.

Limits, Continuity and Differentiability

Let us define,

$$g(x) = \begin{cases} \frac{f(x)-f(\alpha)}{x-\alpha}, & x \neq \alpha \\ f'(\alpha), & x = \alpha \end{cases}$$

Then,  $f(x)-f(\alpha) = (x-\alpha)g(x), \forall x \neq \alpha$ .

Now for continuity of  $g(x)$  at  $x = \alpha$

$$\lim_{x \rightarrow \alpha} g(x) = \lim_{x \rightarrow \alpha} \frac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha) = g(\alpha)$$

$\therefore g$  is continuous at  $x = \alpha$ .

22.

Given that

$$f(x) = \begin{cases} x+a, & \text{if } x < 0 \\ |x-1|, & \text{if } x \geq 0 \end{cases} = \begin{cases} x+a, & \text{if } x < 0 \\ 1-x, & \text{if } 0 \leq x < 1 \\ x-1, & \text{if } x \geq 1 \end{cases}$$

$$\text{and } g(x) = \begin{cases} (x+1), & \text{if } x < 0 \\ (x-1)^2 + b, & \text{if } x \geq 0 \end{cases}$$

where  $a, b \geq 0$

Then  $(g \circ f)(x) = g[f(x)]$

**NOTE THIS STEP**

$$= \begin{cases} f(x)+1, & \text{if } f(x) < 0 \\ [f(x)-1]^2 + b, & \text{if } f(x) \geq 0 \end{cases}$$

(Using definition of  $g(x)$ )

Now,  $f(x) < 0$  when  $x+a < 0$  i.e.  $x < -a$

$f(x) = 0$  when  $x = -a$  or  $x = 1$

$f(x) > 0$  when  $-a < x < 1$  or  $x > 1$

$$g(f(x)) = \begin{cases} f(x)+1, & \text{if } x < -a \\ [f(x)-1]^2 + b, & \text{if } x = -a \text{ or } x = 1 \\ [f(x)-1]^2 + b, & \text{if } -a < x < 1 \\ [f(x)-1]^2 + b, & \text{if } 0 \leq x < 1 \\ [f(x)-1]^2 + b, & \text{if } x > 1 \end{cases}$$

[Keeping in mind that  $x=0$  and  $1$  are also the breaking points because of definition of  $f(x)$ ]

$$\therefore g[f(x)] = \begin{cases} x+a+1, & \text{if } x < -a \\ (x+a-1)^2 + b, & \text{if } -a \leq x < 0 \\ (1+x-1)^2 + b, & \text{if } 0 \leq x \leq 1 \\ (x-1-1)^2 + b, & \text{if } x > 1 \end{cases}$$

Substituting the value of  $f(x)$  under different conditions).

$$\therefore g[f(x)] = \begin{cases} x+a+1, & \text{if } x < -a \\ (x+a-1)^2 + b, & \text{if } -a \leq x < 0 = F(x) \text{ (say)} \\ x^2 + b, & \text{if } 0 \leq x \leq 1 \\ (x-2)^2 + b, & \text{if } x > 1 \end{cases}$$

Now given that  $g \circ f(x) \equiv F(x)$  is continuous for all real numbers, therefore it will be continuous at  $-a$

$$\Rightarrow \text{L.H.S} = \text{R.H.L} = f(-a)$$

$$\lim_{h \rightarrow 0} F(-a-h) = \lim_{h \rightarrow 0} F(-a+h) = f(-a)$$

$$\text{Now, } \lim_{h \rightarrow 0} F(-a-h) = \lim_{h \rightarrow 0} (-a-h+a+1) = 1$$

$$\lim_{h \rightarrow 0} F(-a+h) = \lim_{h \rightarrow 0} (-a+h+a-1)^2 + b = 1+b$$

$$F(-a) = 1+b$$

Thus we should have  $1 = 1+b \Rightarrow b = 0$ .

Again for continuity at  $x = 0$

$$\text{L.H.L} = f(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(0-h) = f(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(-h+a-1)^2 + b = b \Rightarrow (a-1)^2 = 0 \Rightarrow a = 1$$

For  $a = 1$  and  $b = 0$ ,  $g \circ f$  becomes

$$g \circ f(x) = \begin{cases} x+2, & x < -1 \\ x^2, & -1 \leq x \leq 1 \\ (x-2)^2, & x > 1 \end{cases}$$

Now to check differentiability of  $g \circ f(x)$  at  $x = 0$

We see,  $g \circ f(x) = x^2 = F(x)$

$\Rightarrow F'(x) = 2x$  which exists clearly at  $x = 0$ .

$g \circ f$  is differentiable at  $x = 0$

Given that  $f: [-2a, 2a] \rightarrow \mathbb{R}$

$f$  is an odd function.

$Lf'$  at  $x = a$  is 0.

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h} = 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{h} = 0 \quad \dots(1)$$

To find  $Lf'$  at  $x = -a$  which is given by

$$\lim_{h \rightarrow 0} \frac{f(-a-h) - f(-a)}{-h} = \lim_{h \rightarrow 0} \frac{-f(a+h) + f(a)}{-h} \quad [\because f(-x) = -f(x)]$$

$$= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Again for  $x \in [a, 2a]$

$$f(x) = f(2a-x)$$

$$\therefore f(a+h) = f(2a-a-h) = f(a-h)$$

Substituting this values in last expression we get

$$Lf'(-a) = \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{h} = 0 \quad [\text{Using equation (1)}]$$

Hence  $Lf'(-a) = 0$

To find,

$$\lim_{n \rightarrow \infty} \left[ (n+1) \frac{2}{\pi} \cos^{-1} \left( \frac{1}{n} \right) - n \right]$$

$$= \lim_{n \rightarrow \infty} n \left[ \left( 1 + \frac{1}{n} \right) \frac{2}{\pi} \cos^{-1} \left( \frac{1}{n} \right) - 1 \right] = \lim_{n \rightarrow \infty} n f \left( \frac{1}{n} \right)$$

where  $f(x) = \left[ (1+x) \frac{2}{\pi} \cos^{-1} x - 1 \right]$  such that

$$f(0) = \left[ (1+0) \frac{2}{\pi} \cos^{-1} 0 - 1 \right] = \frac{2}{\pi} \cdot \frac{\pi}{2} - 1 = 0$$

$\therefore$  Using given relation as  $\lim_{n \rightarrow \infty} n f\left(\frac{1}{n}\right) = f'(0)$

then given limit becomes

$$= f'(0) = \frac{d}{dx} \left[ (1+x) \frac{2}{\pi} \cos^{-1} x - 1 \right] \Bigg|_{x=0}$$

$$= \frac{2}{\pi} \left[ \cos^{-1} x - \frac{1-x}{\sqrt{1-x^2}} \right] \Bigg|_{x=0}$$

$$= \frac{2}{\pi} \left[ \frac{\pi}{2} - 1 \right] = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}$$

25.

Given that,  $f(x)$  is differentiable at  $x=0$ .

Hence,  $f(x)$  will also be continuous at  $x=0$

$$\Rightarrow \lim_{h \rightarrow 0} f(0+h) = f(0) \Rightarrow \lim_{h \rightarrow 0} \frac{e^{\frac{ah}{2}} - 1}{h} = \frac{1}{2}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{e^{\frac{ah}{2}} - 1}{\frac{ah}{2}} \times \frac{a}{2} = \frac{1}{2} \Rightarrow a = 1$$

Also differentiability of  $f(x)$  at  $x=0$ , gives

$$Lf'(0) = Rf'(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{b \sin^{-1} \left( \frac{c-h}{2} \right) - \frac{1}{2}}{-h} = \lim_{h \rightarrow 0} \frac{e^{\frac{ah}{2}} - 1 - \frac{1}{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2e^{\frac{ah}{2}} - 2 - h}{2h^2} \quad \left[ \text{form } \frac{0}{0} \right]$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{1 - \left( \frac{c-h}{2} \right)^2} \cdot \left( -\frac{1}{2} \right)}{-1} \quad \left[ \text{Using L Hospital's rule} \right]$$

$$= \lim_{h \rightarrow 0} \frac{2e^{\frac{ah}{2}} \cdot \frac{a}{2} - 1}{4h} = \lim_{h \rightarrow 0} \frac{e^{\frac{h}{2}} - 1}{8 \left( \frac{h}{2} \right)} \quad \left[ \text{Putting } a = 1 \right]$$

$$\Rightarrow \frac{b}{\sqrt{1 - \frac{c^2}{4}}} = \frac{1}{8} \Rightarrow 4b = \sqrt{1 - \frac{c^2}{4}} \Rightarrow 16b^2 = \frac{4 - c^2}{4}$$

$$\Rightarrow 64b^2 = 4 - c^2 \quad \text{Hence proved.}$$

26.

Given that,

$$f(x-y) = f(x) \cdot g(y) - f(y) \cdot g(x) \quad \dots(i)$$

$$g(x-y) = g(x) \cdot g(y) + f(x)f(y) \quad \dots(ii)$$

In eqn. (i), putting  $x=y$ , we get

$$f(0) = f(x)g(x) - f(x)g(x) \Rightarrow f(0) = 0$$

Putting  $y=0$ , in eqn. (i), we get

$$f(x) = f(x)g(0) - f(0)g(x)$$

$$\Rightarrow f(x) = f(x)g(0) \quad \left[ \text{using } f(0) = 0 \right]$$

$$\Rightarrow g(0) = 1$$

Putting  $x=y$  in eqn. (ii), we get

$$g(0) = g(x)g(x) + f(x)f(x)$$

$$\Rightarrow 1 = [g(x)]^2 + [f(x)]^2 \quad \left[ \text{using } g(0) = 1 \right]$$

$$\Rightarrow [g(x)]^2 = 1 - [f(x)]^2 \quad \dots(iii)$$

Clearly  $g(x)$  will be differentiable only if  $f(x)$  is differentiable.

$\therefore$  First we will check the differentiability of  $f(x)$

Given that  $Rf'(0)$  exists

$$\text{i.e., } \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \text{ exists}$$

$$\text{i.e., } \lim_{h \rightarrow 0} \frac{f(0)g(-h) - f(-h)g(0)}{h} \text{ exists}$$

$$\text{i.e., } \lim_{h \rightarrow 0} \frac{-f(-h)}{h} \text{ exists (using } f(0) = 0 \text{ and } g(0) = 1)$$

Which can be written as,

$$\lim_{h \rightarrow 0} \frac{f(0) - f(-h)}{-h} = Lf'(0)$$

$$\Rightarrow Lf'(0) = Rf'(0)$$

$\therefore f$  is differentiable, at  $x=0$

Differentiating equation (iii), we get

$$2g(x) \cdot g'(x) = -2f(x) \cdot f'(x)$$

For  $x=0$

$$\Rightarrow g(0) \cdot g'(0) = -f(0) \cdot f'(0)$$

$$\Rightarrow g'(0) = 0 \quad \left[ \text{Using } f(0) = 0 \text{ and } g(0) = 1 \right]$$

### F. Match the Following

1. (A)  $\sin(\pi[x]) = 0, \forall x \in R$

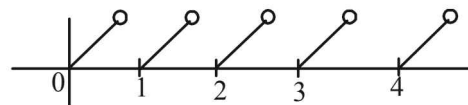
$\therefore$  Differentiable everywhere.

$\therefore$  (A)  $\rightarrow$  (p)

(B)  $\sin(\pi(x - [x])) = f(x)$

We know that

$$x - [x] = \begin{cases} = x, & \text{if } 0 \leq x < 1 \\ = x - 1, & \text{if } 1 \leq x < 2 \\ = x - 2, & \text{if } 2 \leq x < 3 \end{cases}$$



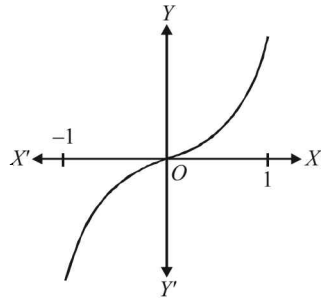
Its graph is, as shown in figure which is discontinuous at  $\forall x \in Z$ . Clearly  $x - [x]$  and hence  $\sin(\pi(x - [x]))$  is not differentiable  $\forall x \in Z$

(B)  $\rightarrow$  r

2. (A)  $y = x|x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

Limits, Continuity and Differentiability

Graph is as follows :

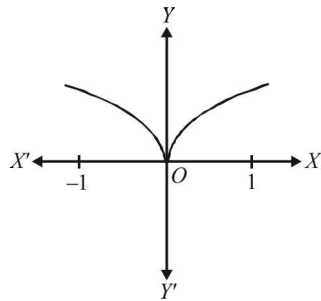


From graph  $y = x|x|$  is continuous in  $(-1, 1)$  (p)  
 differentiable in  $(-1, 1)$  (q)  
 Strictly increasing in  $(-1, 1)$ . (r)

(B)  $y = \sqrt{|x|} = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$   
 $\Rightarrow y^2 = -x, x < 0$   
 {where  $y$  can take only + ve values}

and  $y^2 = x, x \geq 0$

$\therefore$  Graph is as follows :

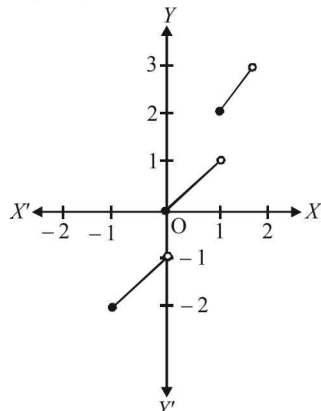


From graph  $y = \sqrt{|x|}$  is continuous in  $(-1, 1)$  (p)  
 not differentiable at  $x = 0$  (s)

(C) NOTE THIS STEP

$$y = x + [x] = \begin{cases} - & - & - \\ x-1, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ x+1, & 1 \leq x < 2 \\ - & - & - \end{cases}$$

$\therefore$  Graph of  $y = x + [x]$  is as follows :

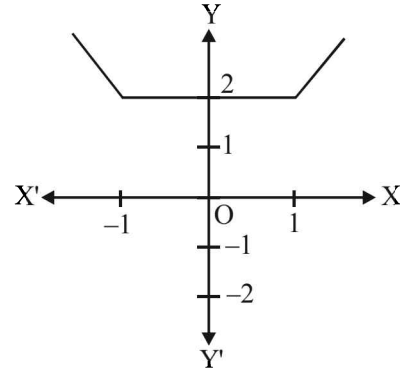


From graph,  $y = x + [x]$  is neither continuous, nor differentiable at  $x = 0$  and hence in  $(-1, 1)$ . (s)

Also it is strictly increasing in  $(-1, 1)$  (r)

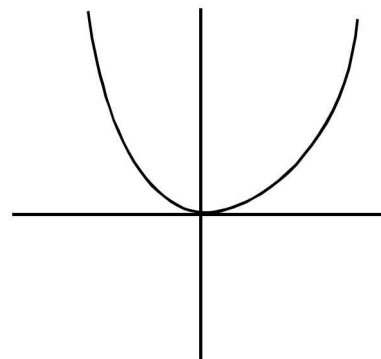
(D)  $y = |x-1| + |x+1| = \begin{cases} -2x, & x < -1 \\ 2, & -1 \leq x < 1 \\ 2x, & x \geq 1 \end{cases}$

Graph of function is as follows :



From graph,  $y = f(x)$  is continuous (p) and differentiable (q) in  $(-1, 1)$  but not strictly increasing in  $(-1, 1)$ .

3. (d)  $P(1): f_4(x) = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \geq 0 \end{cases}$

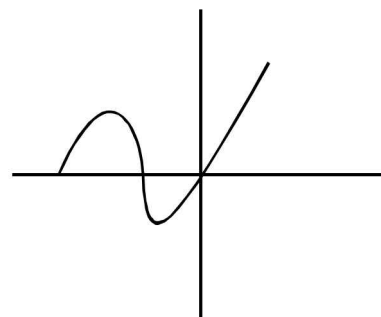


Range of  $f_4 = [0, \infty)$

$\therefore f_4$  is onto

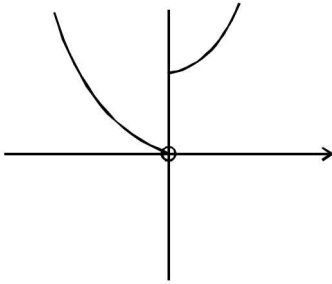
From graph  $f_4$  is not one one.

$Q(3): f_3(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \geq 0 \end{cases}$



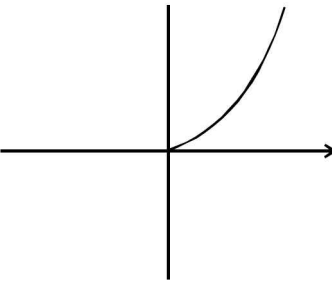
From graph  $f$  is differentiable but not one one.

$R(2): f_2 \circ f_1(x) = \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \geq 0 \end{cases}$



From graph  $f_2 \circ f_1$  is neither continuous nor one one.

S(4):  $f_2(x) = x^2, x \in [0, \infty)$



It is continuous and one one.

**I. Integer Value Correct Type**

1. (6)

$$6 \int_1^x f(t) dt = 3xf(x) - x^3$$

Differentiating, we get  $6f(x) = 3f(x) + 3xf'(x) - 3x^2$

$$\Rightarrow f'(x) - \frac{1}{x}f(x) = x$$

$$\text{I.F.} = \frac{1}{x}$$

$$\therefore \text{Solution is } f(x) \cdot \frac{1}{x} = \int 1 \cdot dx = x + c$$

$$\therefore f(x) = x^2 + cx$$

$$\text{But } f(1) = 2 \Rightarrow c = 1 \therefore f(x) = x^2 + x$$

$$\text{Hence } f(2) = 4 + 2 = 6$$

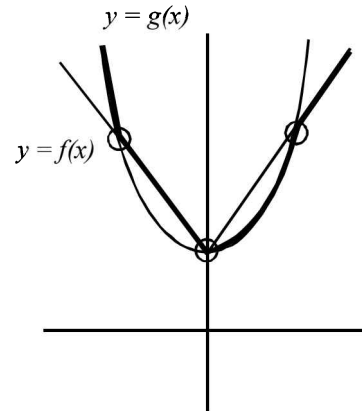
**Note :** Putting  $x = 1$  in given integral equation, we get

$$f(1) = \frac{1}{3} \text{ while given } f(1) = 2.$$

$\therefore$  Data given in the question is inconsistent.

2. (2) 
$$\lim_{x \rightarrow 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{1-\sqrt{x}} = \frac{1}{4}$$
- $$\Rightarrow \lim_{x \rightarrow 1} \left\{ \frac{a(1-x) + \sin(x-1)}{(x-1) + \sin(x-1)} \right\}^{1+\sqrt{x}}$$
- $$\Rightarrow \lim_{x \rightarrow 1} \left\{ \frac{-a + \frac{\sin(x-1)}{x-1}}{1 + \frac{\sin(x-1)}{x-1}} \right\}^{1+\sqrt{x}} \Rightarrow \left( \frac{-a+1}{2} \right)^2 = \frac{1}{4}$$
- $\Rightarrow a = 0$  or  $2$   
 $\therefore$  Largest value of  $a$  is  $2$ .

3. (3) 
$$f(x) = |x| + 1 = \begin{cases} x+1, & x \geq 0 \\ -x+1, & x < 0 \end{cases}$$
- $$g(x) = x^2 + 1$$



From graph there are 3 points at which  $h(x)$  is not differentiable.

4. (2) 
$$\lim_{\alpha \rightarrow 0} \frac{e^{\cos \alpha^n} - e}{\alpha^m} = \frac{-e}{2}$$
- $$\Rightarrow \lim_{\alpha \rightarrow 0} \frac{e^{[\cos \alpha^n - 1]} - 1}{\cos \alpha^n - 1} \times \frac{\cos \alpha^n - 1}{\alpha^m} = \frac{-e}{2}$$
- $$\Rightarrow e \lim_{\alpha \rightarrow 0} \frac{-2 \sin^2 \frac{\alpha^n}{2}}{\left(\frac{\alpha^n}{2}\right)^2} \times \frac{\left(\frac{\alpha^n}{2}\right)^2}{\alpha^m} = \frac{-e}{2}$$
- $$\Rightarrow \frac{-e}{2} \alpha^{2n-m} = \frac{-e}{2} \text{ or } \alpha^{2n-m} = 1$$
- $$\Rightarrow 2n - m = 0 \Rightarrow \frac{m}{n} = 2$$
5. (7) 
$$\lim_{x \rightarrow 0} \frac{x^2 \sin \beta x}{\alpha x - \sin x} = 1$$
- $$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \beta}{\alpha x - \sin x} = 1$$
- $$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \beta}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \infty\right)} = 1$$
- $$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \beta}{(\alpha - 1)x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots \infty} = 1$$
- For above to be possible, we should have
- $$\alpha - 1 = 0 \text{ and } \beta = \frac{1}{3!}$$
- $$\Rightarrow \alpha = 1 \text{ and } \beta = \frac{1}{6}$$
- $$\therefore 6(\alpha + \beta) = 6\left(1 + \frac{1}{6}\right) = 7$$

Section-B

JEE Main/ AIEEE

1. (d)  $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2x}} \Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1 - (1 - 2 \sin^2 x)}}{\sqrt{2x}}$ ;  
 $\lim_{x \rightarrow 0} \frac{\sqrt{2 \sin^2 x}}{\sqrt{2x}} \Rightarrow \lim_{x \rightarrow 0} \frac{|\sin x|}{x}$

The limit of above does not exist as  
 LHS = -1 ≠ RHL = 1

2. (a)  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + 5x + 3}{x^2 + x + 2} \right)^x = \lim_{x \rightarrow \infty} \left( 1 + \frac{4x + 1}{x^2 + x + 2} \right)^x$   
 $= \lim_{x \rightarrow \infty} \left[ \left( 1 + \frac{4x + 1}{x^2 + x + 2} \right)^{\frac{x^2 + x + 2}{4x + 1}} \right]^{\frac{(4x + 1)x}{x^2 + x + 2}}$   
 $= e^{\lim_{x \rightarrow \infty} \frac{4x^2 + x}{x^2 + x + 2} \left[ \lim_{x \rightarrow \infty} (1 + \lambda x)^{\frac{1}{x}} = e^\lambda \right]}$   
 $= e^{\lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x}}{1 + \frac{1}{x} + \frac{2}{x^2}}} = e^4$

3. (c) Apply L H Rule

We have,  $\lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x - 2} \left( \frac{0}{0} \right)$   
 $= \lim_{x \rightarrow 2} f(2) - 2f'(x) = f(2) - 2f'(2)$   
 $= 4 - 2 \times 4 = -4.$

4. (a) We have  $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}$ ;

$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^p}{n^p \cdot n} = \int_0^1 x^p dx = \left[ \frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{p+1}$

5. (d) Since  $\lim_{x \rightarrow 0} [x]$  does not exist, hence the required limit does not exist.

6. (a)  $\lim_{x \rightarrow 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1} \left( \frac{0}{0} \right)$  form using L' Hospital's rule  
 $= \lim_{x \rightarrow 1} \frac{\frac{1}{2\sqrt{f(x)}} f'(x)}{1/2\sqrt{x}} = \frac{f'(1)}{\sqrt{f(1)}} = \frac{2}{1} = 2.$

7. (b) Let  $a$  is a rational number other than 0, in  $[-5, 5]$ , then

$f(a) = a$  and  $\lim_{x \rightarrow a} f(x) = -a$

[As in the immediate neighbourhood of a rational number, we find irrational numbers]

∴  $f(x)$  is not continuous at any rational number

If  $a$  is irrational number, then

$f(a) = -a$  and  $\lim_{x \rightarrow a} f(x) = a$

∴  $f(x)$  is not continuous at any irrational number clearly

$\lim_{x \rightarrow 0} f(x) = f(0) = 0$

∴  $f(x)$  is continuous at  $x = 0$

8. (d) ∴  $f''(x) - g''(x) = 0$

Integrating,  $f'(x) - g'(x) = c$ ;

$\Rightarrow f'(1) - g'(1) = c \Rightarrow 4 - 2 = c \Rightarrow c = 2.$

∴  $f'(x) - g'(x) = 2$ ;

Integrating,  $f(x) - g(x) = 2x + c_1$

$\Rightarrow f(2) - g(2) = 4 + c_1 \Rightarrow 9 - 3 = 4 + c_1$ ;

$\Rightarrow c_1 = 2 \quad \therefore f(x) - g(x) = 2x + 2$

At  $x = 3/2, f(x) - g(x) = 3 + 2 = 5.$

9. (c)  $f(x+y) = f(x) \times f(y)$

Differentiate with respect to  $x$ , treating  $y$  as constant

$f'(x+y) = f'(x)f(y)$

Putting  $x = 0$  and  $y = x$ , we get  $f'(x) = f'(0)f(x)$ ;

$\Rightarrow f'(5) = 3f(5) = 3 \times 2 = 6.$

10. (a) The given expression can be written as

$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left( \frac{r}{n} \right)^4 - \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{r}{n} \right)^3$

$= \int_0^1 x^4 dx - \lim_{n \rightarrow \infty} \frac{1}{n} \times \int_0^1 x^3 dx = \left[ \frac{x^5}{5} \right]_0^1 - 0 = \frac{1}{5}$

11. (d)  $\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = K$  (by L'Hospital rule)

$\lim_{x \rightarrow 0} \frac{\frac{1}{3+x} - \frac{-1}{3-x}}{1} = K \quad \therefore \frac{2}{3} = K$

12. (d)  $\lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^{x^2} \sec^2 t dt}{\frac{d}{dx} (x \sin x)} = \lim_{x \rightarrow 0} \frac{\sec^2 x^2 \cdot 2x}{\sin x + x \cos x}$

(by L' Hospital rule)

$\lim_{x \rightarrow 0} \frac{2 \sec^2 x^2}{\left( \frac{\sin x}{x} + \cos x \right)} = \frac{2 \times 1}{1 + 1} = 1$

13. (b)  $\lim_{x \rightarrow a} \frac{f(a)g'(x) - g(a)f'(x)}{g'(x) - f'(x)} = 4$

(By L' Hospital rule)

$\lim_{x \rightarrow a} \frac{k g'(x) - k f'(x)}{g'(x) - f'(x)} = 4 \quad \therefore k = 4.$



14. (d) 
$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan\left(\frac{\pi-x}{4}\right) \cdot (1-\sin x)}{(\pi-2x)^3}$$

$$\text{Let } x = \frac{\pi}{2} + y; y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{\tan\left(-\frac{y}{2}\right) \cdot (1-\cos y)}{(-2y)^3} = \lim_{y \rightarrow 0} \frac{-\tan\frac{y}{2} \cdot 2\sin^2\frac{y}{2}}{(-8) \cdot \frac{y^3}{8}}$$

$$= \lim_{y \rightarrow 0} \frac{1}{32} \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)^2} \cdot \left[\frac{\sin y/2}{y/2}\right]^2 = \frac{1}{32}$$

15. (c)  $f(0) = 0; f(x) = xe^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}$

R.H.L.  $\lim_{h \rightarrow 0} (0+h)e^{-2/h} = \lim_{h \rightarrow 0} \frac{h}{e^{2/h}} = 0$

L.H.L.  $\lim_{h \rightarrow 0} (0-h)e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} = 0$

therefore,  $f(x)$  is continuous.

R.H.D. 
$$\lim_{h \rightarrow 0} \frac{(0+h)e^{-\left(\frac{1}{h} + \frac{1}{h}\right)} - 0}{h} = 0$$

L.H.D. 
$$\lim_{h \rightarrow 0} \frac{(0-h)e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} - 0}{-h} = 1$$

therefore, L.H.D.  $\neq$  R.H.D.  
 $f(x)$  is not differentiable at  $x = 0$ .

16. (b) We know that  $\lim_{x \rightarrow \infty} (1+x)^{\frac{1}{x}} = e$

$\therefore \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$

$$\Rightarrow \lim_{x \rightarrow \infty} \left[ \left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{\left(\frac{1}{x} + \frac{b}{x^2}\right)} \right]^{2x \left(\frac{a+b}{x+x^2}\right)} = e^2$$

$$\Rightarrow \lim_{x \rightarrow \infty} 2 \left[ \frac{a+b}{x} \right] = e^2 \Rightarrow e^{2a} = e^2 \Rightarrow a = 1 \text{ and } b \in R$$

17. (c)  $f(x) = \frac{1-\tan x}{4x-\pi}$  is continuous in  $\left[0, \frac{\pi}{2}\right]$

$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x \rightarrow \frac{\pi}{4}} f(x) = \lim_{x \rightarrow \frac{\pi}{4}} f(x) = \lim_{h \rightarrow 0} f\left(\frac{\pi}{4} + h\right)$

$$= \lim_{h \rightarrow 0} \frac{1-\tan\left(\frac{\pi}{4} + h\right)}{4\left(\frac{\pi}{4} + h\right) - \pi}, h > 0 = \lim_{h \rightarrow 0} \frac{1-\frac{1+\tan h}{1-\tan h}}{4h}$$

$$= \lim_{h \rightarrow 0} \frac{-2}{1-\tanh} \cdot \frac{\tan h}{4h} = \frac{-2}{4} = -\frac{1}{2}$$

18. (d) 
$$\lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \frac{3}{n^2} \sec^2 \frac{9}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right]$$

$$= \lim_{n \rightarrow \infty} \frac{r}{n^2} \sec^2 \frac{r^2}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{r}{n} \sec^2 \frac{r^2}{n^2}$$

$\Rightarrow$  Given limit is equal to value of integral

$$\int_0^1 x \sec^2 x^2 dx$$

or  $\frac{1}{2} \int_0^1 2x \sec^2 x^2 dx = \frac{1}{2} \int_0^1 \sec^2 t dt$  [put  $x^2 = t$ ]

$$= \frac{1}{2} (\tan t)_0^1 = \frac{1}{2} \tan 1.$$

19. (a) Given limit 
$$= \lim_{x \rightarrow \alpha} \frac{1 - \cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2}$$

$$= \lim_{x \rightarrow \alpha} \frac{2 \sin^2 \left( a \frac{(x-\alpha)(x-\beta)}{2} \right)}{(x-\alpha)^2}$$

$$= \lim_{x \rightarrow \alpha} \frac{2}{(x-\alpha)^2} \times \frac{\sin^2 \left( a \frac{(x-\alpha)(x-\beta)}{2} \right)}{a^2 (x-\alpha)^2 (x-\beta)^2}$$

$$= \frac{a^2 (\alpha-\beta)^2}{2} \times \frac{a^2 (x-\alpha)^2 (x-\beta)^2}{4}$$

20. (c)  $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$ ;

As function is differentiable so it is continuous as it is given that  $\lim_{h \rightarrow 0} \frac{f(1+h)}{h} = 5$  and hence  $f(1) = 0$

Hence  $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h)}{h} = 5$

21. (a) As  $f(1) = -2$  &  $f'(x) \geq 2 \forall x \in [1, 6]$

Applying Lagrange's mean value theorem

$$\frac{f(6) - f(1)}{5} = f'(c) \geq 2 \Rightarrow f(6) \geq 10 + f(1)$$

$$\Rightarrow f(6) \geq 10 - 2 \Rightarrow f(6) \geq 8.$$

22. (b)  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$|f'(x)| = \lim_{h \rightarrow 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq \lim_{h \rightarrow 0} \left| \frac{(h)^2}{h} \right|$$

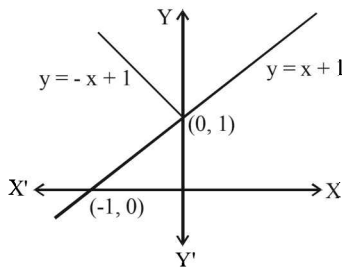
$$\Rightarrow |f'(x)| \leq 0 \Rightarrow f'(x) = 0$$

$$\Rightarrow f(x) = \text{constant}$$

As  $f(0) = 0 \Rightarrow f(1) = 0$

## Limits, Continuity and Differentiability

23. (a)  $f(x) = \min \{x+1, |x|+1\} \Rightarrow f(x) = x+1 \quad \forall x \in R$



Hence,  $f(x)$  is differentiable everywhere for all  $x \in R$ .

24. (b) Given,  $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1} \Rightarrow f(0) = \lim_{x \rightarrow 0} \frac{1}{x} - \frac{2}{e^{2x}-1}$

$$= \lim_{x \rightarrow 0} \frac{(e^{2x}-1)-2x}{x(e^{2x}-1)} \quad \left[ \frac{0}{0} \text{ form} \right]$$

$\therefore$  using, L'Hospital rule

$$f(0) = \lim_{x \rightarrow 0} \frac{4e^{2x}}{2(xe^{2x} + e^{2x}) + e^{2x} \cdot 2}$$

$$= \lim_{x \rightarrow 0} \frac{4e^{2x}}{4xe^{2x} + 2e^{2x} + 2e^{2x}} \quad \left[ \frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{4e^{2x}}{4(xe^{2x} + e^{2x})} = \frac{4 \cdot e^0}{4(0 + e^0)} = 1$$

25. (c) We have  $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$

$$Rf'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h} = \text{a finite number}$$

Let this finite number be  $l$

$$Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-h \sin\left(\frac{1}{-h}\right)}{-h} = \lim_{h \rightarrow 0} \sin\left(\frac{1}{-h}\right)$$

$$= -\lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) = -(\text{a finite number}) = -l$$

Thus  $Rf'(1) \neq Lf'(1)$

$\therefore f$  is not differentiable at  $x = 1$

$$\text{Also, } f'(0) = \sin \frac{1}{(x-1)} - \frac{x-1}{(x-1)^2} \cos\left(\frac{1}{x-1}\right) \Bigg|_{x=0}$$

$$= -\sin 1 + \cos 1 \quad \therefore f \text{ is differentiable at } x = 0$$

26. (d)  $f(x)$  is a positive increasing function.

$$\therefore 0 < f(x) < f(2x) < f(3x)$$

$$\Rightarrow 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} 1 \leq \lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} \leq \lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)}$$

By Sandwich Theorem.

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} = 1$$

27. (d)  $\lim_{x \rightarrow 2} \frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2} = \lim_{x \rightarrow 2} \frac{\sqrt{2} |\sin(x-2)|}{x-2}$

$$\text{L.H.L.} = \lim_{x \rightarrow 2^-} \frac{\sqrt{2} \sin(x-2)}{(x-2)} = -\sqrt{2}$$

$$\text{R.H.L.} = \lim_{x \rightarrow 2^+} \frac{\sqrt{2} \sin(x-2)}{(x-2)} = \sqrt{2}$$

Thus L.H.L.  $\neq$  R.H.L.

Hence,  $\lim_{x \rightarrow 2} \frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2}$  does not exist.

28. (b)  $L.H.L. = \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{\sin\{(p+1)(-h)\} - \sin(-h)}{-h}$

$$= \lim_{h \rightarrow 0} \frac{-\sin(p+1)h}{-h} + \frac{\sin(-h)}{-h} = p+1+1 = p+2$$

$$R.H.L. = \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{(\sqrt{1+h}+1)} = \frac{1}{2}$$

$$\text{and } f(0) = q \Rightarrow p = -\frac{3}{2}, q = \frac{1}{2}$$

29. (d)  $\lim_{x \rightarrow 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$

$$\lim_{x \rightarrow 5} [(f(x))^2 - 9] = 0 \Rightarrow \lim_{x \rightarrow 5} f(x) = 3$$

30. (a) Let  $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)$

Doubtful points are  $x = n, n \in I$

$$\begin{aligned} \text{L.H.L.} &= \lim_{x \rightarrow n^-} [x] \cos\left(\frac{2x-1}{2}\right) \pi \\ &= (n-1) \cos\left(\frac{2n-1}{2}\right) \pi = 0 \end{aligned}$$

( $\because [x]$  is the greatest integer function)

$$\text{R.H.L.} = \lim_{x \rightarrow n^+} [x] \cos\left(\frac{2x-1}{2}\right) \pi = n \cos\left(\frac{2n-1}{2}\right) \pi = 0$$

Now, value of the function at  $x = n$  is  $f(n) = 0$

Since, L.H.L. = R.H.L. =  $f(n)$

$\therefore f(x) = [x] \cos\left(\frac{2x-1}{2}\right)$  is continuous for every real  $x$ .

31. (c)  $f(x) = |x-2| = \begin{cases} x-2, & x-2 \geq 0 \\ 2-x, & x-2 \leq 0 \end{cases}$

$$= \begin{cases} x-2, & x \geq 2 \\ 2-x, & x \leq 2 \end{cases}$$

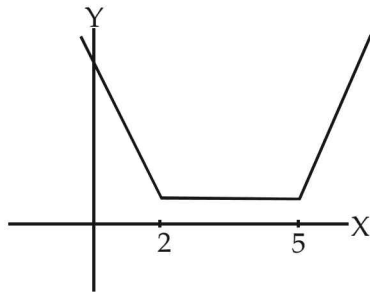
Similarly,  $f(x) = |x-5| = \begin{cases} x-5, & x \geq 5 \\ 5-x, & x \leq 5 \end{cases}$

$\therefore f(x) = |x-2| + |x-5| = \{x-2+5-x = 3, 2 \leq x \leq 5\}$

Thus  $f(x) = 3, 2 \leq x \leq 5$

$f'(x) = 0, 2 < x < 5$

$f'(4) = 0 \therefore$  statement 1 is true.



$\therefore f(2) = 0 + |2-5| = 3$  and  $f(5) = |5-2| + 0 = 3$

$\therefore$  statement-2 is also true and a correct explanation for statement 1.

32. (d) Multiply and divide by  $x$  in the given expression, we get

$$\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x^2} \cdot \frac{x}{\tan 4x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \rightarrow 0} 3 + \cos x \cdot \lim_{x \rightarrow 0} \frac{x}{\tan 4x}$$

$$= 2.4 \cdot \frac{1}{4} \lim_{x \rightarrow 0} \frac{4x}{\tan 4x} = 2.4 \cdot \frac{1}{4} = 2$$

33. (b) Consider  $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{\sin[\pi(1 - \sin^2 x)]}{x^2}$$

$$= \lim_{x \rightarrow 0} \sin \frac{(\pi - \pi \sin^2 x)}{x^2} \quad [\because \sin(\pi - \theta) = \sin \theta]$$

$$= \lim_{x \rightarrow 0} \sin \frac{(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi \sin^2 x}{x^2}$$

$$= \lim_{x \rightarrow 0} 1 \times \pi \left(\frac{\sin x}{x}\right)^2 = \pi$$

34. (a) Multiply and divide by  $x$  in the given expression, we get

$$\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x^2} \cdot \frac{x}{\tan 4x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \rightarrow 0} 3 + \cos x \cdot \lim_{x \rightarrow 0} \frac{x}{\tan 4x}$$

$$= 2.4 \cdot \frac{1}{4} \lim_{x \rightarrow 0} \frac{4x}{\tan 4x} = 2.4 \cdot \frac{1}{4} = 2$$

35. (c) Since  $g(x)$  is differentiable, it will be continuous at  $x=3$

$$\therefore \lim_{x \rightarrow 3^-} g(x) = \lim_{x \rightarrow 3^+} g(x)$$

$$2k = 3m + 2 \quad \dots(1)$$

Also  $g(x)$  is differentiable at  $x = 0$

$$\therefore \lim_{x \rightarrow 3^-} g'(x) = \lim_{x \rightarrow 3^+} g'(x)$$

$$\frac{K}{2\sqrt{3+1}} = m$$

$$k = 4m \quad \dots(2)$$

Solving (1) and (2), we get

$$m = \frac{2}{5}, k = \frac{8}{5}$$

$$\therefore k + m = 2$$

36. (d)  $g(x) = f(f(x))$

In the neighbourhood of  $x = 0$ ,

$$f(x) = |\log 2 - \sin x| = (\log 2 - \sin x)$$

$$\therefore g(x) = |\log 2 - \sin| \log 2 - \sin x ||$$

$$= (\log 2 - \sin(\log 2 - \sin x))$$

$\therefore g(x)$  is differentiable

$$\text{and } g'(x) = -\cos(\log 2 - \sin x) (-\cos x)$$

$$\Rightarrow g'(0) = \cos(\log 2)$$

$$37. \text{ (d) } y = \lim_{n \rightarrow \infty} \left( \frac{(n+1)(n+2)\dots 3n}{n^{2n}} \right)^{\frac{1}{n}}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( 1 + \frac{1}{n} \right) \left( 1 + \frac{2}{n} \right) \dots \left( 1 + \frac{2n}{n} \right)$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \ln \left( 1 + \frac{1}{n} \right) + \ln \left( 1 + \frac{2}{n} \right) + \dots + \ln \left( 1 + \frac{2n}{n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \ln \left( 1 + \frac{r}{n} \right)$$

$$= \int_0^2 \ln(1+x) dx$$

$$\text{Let } 1+x=t \Rightarrow dx=dt$$

$$\text{when } x=0, t=1$$

$$x=2, t=3$$

$$\ln y = \int_1^3 \ln t dt = [t \ln t - t]_1^3 = \ln \left( \frac{3^3}{e^2} \right) = \ln \left( \frac{27}{e^2} \right)$$

$$\Rightarrow y = \frac{27}{e^2}$$

$$38. \text{ (a) } \ln P = \lim_{x \rightarrow 0^+} \frac{1}{2x} \ln(1 + \tan^2 \sqrt{x})$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \ln(\sec \sqrt{x})$$

Applying L hospital's rule :

$$= \lim_{x \rightarrow 0^+} \frac{\sec \sqrt{x} \tan \sqrt{x}}{\sec \sqrt{x} \cdot 2\sqrt{x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\tan \sqrt{x}}{2\sqrt{x}}$$

$$= \frac{1}{2}$$

